Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

4/8/2020: First hourly Practice A

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. Hand-write your paper. It can be handwritten also electronically using an ipad but it needs to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ 20: False. Also, please sign your solutions.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper.
- The exam will be released at 9 AM on April 8. Try to do it during regular class time. You have to submit the paper within 24 hours by emailing it to knill@math.harvard.edu as an attachment. Submit it soon so that technical problems can be ironed out. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) **T** F For any continuous function f we have $\int_0^1 3f(t)\,dt = 3\int_0^1 f(t)\,dt$.

Solution:
Yes this is linearity.

2) T **F** For any continuous function $\int_0^3 f(t)\,dt = 3\int_0^1 f(t)\,dt$.

Solution:
Looks good but is nonsense.

3) **T** F For any continuous function $\int_0^1 1 - f(t)\,dt = 1 - (\int_0^1 f(t)\,dt)$.

Solution:
Because the integral over 1 can be computed directly.

4) **T** F The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

Solution:
Differentiate the right hand side to check.

5) **T** F The fundamental theorem of calculus implies that $\int_1^3 f'(x)\,dx = f(3) - f(1)$.

Solution:
Yes, this is it.

6) **T** F The integral $\pi\int_0^1 x^2\,dx$ gives the volume of a cone of height 1.

Solution:
Yes the area of a slice is $x^2\pi$.

7) T F The anti-derivative of $1/\cos^2(x)$ is $\tan(x)$.

Solution:
Check.

8) T F The function $F(x) = \int_0^x \tan(t^2)\, dt$ has the derivative $\tan(x^2)$.

Solution:
The first derivative of F is f.

9) T F The function $f(x) = \sin(x)/2$ on $[0, \pi]$ and $f(x) = 0$ else is a PDF

Solution:
We have seen this in class.

10) T F The identity $\frac{d}{dx} \int_1^2 \log(x)\, dx = \log(2) - \log(1)$ holds.

Solution:
We differentiate a constant.

11) T F The function $f(x) = e^{-x^2/2}$ is called the normal distribution.

Solution:
The constant is off.

12) T F If $f < 1$, then $\int_0^2 f(x)\, dx$ can be bigger than 1.

Solution:
Take $f(x) = 0.6$ for example.

13) T **F** An improper integral is an improperly defined definite indefinite integral.

Solution:
If you marked this true, you must have been properly drunk or behaved improperly.

14) **T** F The anti derivative $F(x)$ of $f(x)$ satisfies $F'(x) = f(x)$.

Solution:
This is the fundamental theorem of calculus

15) **T** F A parameter value c for which the number of minima are different for parameters smaller or larger than c is called a catastrophe.

Solution:
This is a definition.

16) T **F** If f is unbounded at 0, then $\int_0^1 f(x)\, dx$ is infinite.

Solution:
The function $\sqrt{x}$ was a counter example.

17) T **F** If $f(-1) = 0$ and $f(1) = 1$ then $f' = 2$ somewhere on $(-1, 1)$.

Solution:
This is close to the intermediate value theorem.

18) **T** F The anti-derivative of $\log(x)$ is $x \log(x) - x + C$, where log is the natural log.

Solution:
You might not have known this by heart, but you can check it!

19) T F The sum $\frac{1}{n}[(\frac{0}{n})^2 + (\frac{1}{n})^2 + ... + (\frac{n-1}{n})^2]$ converges to $1/3$ in the limit $n \to \infty$.

Solution:
It is a Riemann sum.

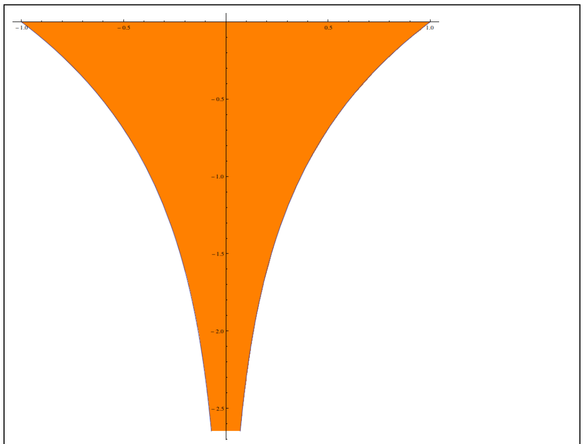
20) T F The **improper integral** $\int_1^\infty \frac{1}{x^2} \, dx$ represents a finite area.

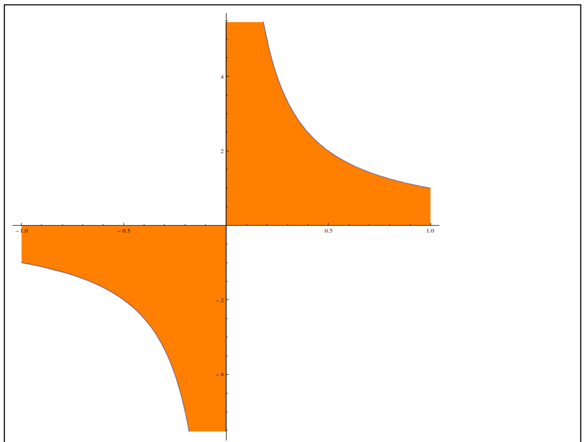
Solution:
We have no problem at infinity.

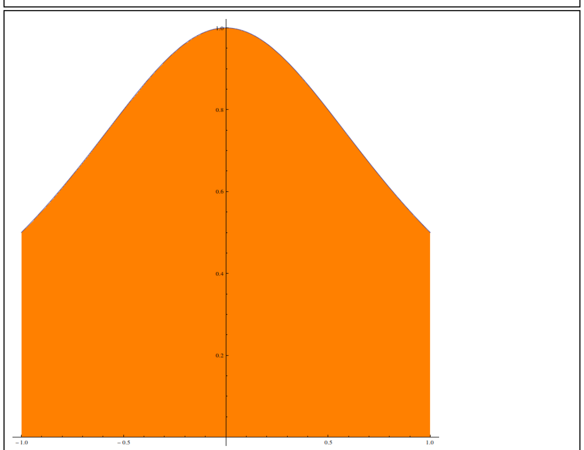
Problem 2) Matching problem (10 points) No justifications are needed.

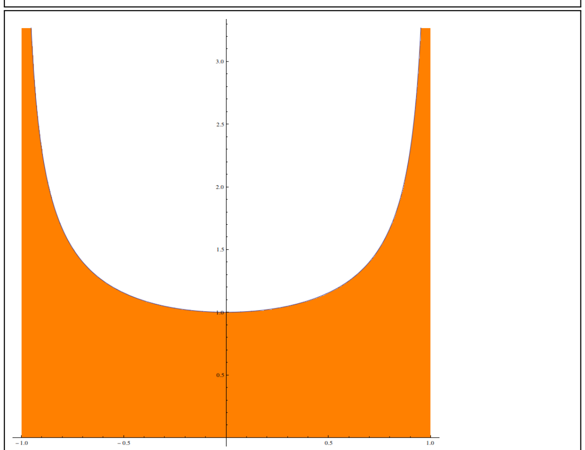
a) (4 points) Match the following integrals with the regions and indicate whether the integral represents a finite area.

Integral	Fill in 1-4	Finite?
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx$		
$\int_{-1}^{1} \frac{1}{x} \, dx$		
$\int_{-1}^{1} \frac{1}{1+x^2} \, dx$		
$\int_{-1}^{1} \log \lvert x \rvert \, dx$		


1)


2)


3)


4)

b) (6 points) Which of the following properties are always true. This means which are true for all choices of continuous functions and all choices of a, b, c.

Identity	Check if true
$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$	
$\int_a^b f(x)\ dx + \int_a^b g(x)\ dx = \int_a^b f(x) + g(x)\ dx$	
$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$	
$\int_a^b f(x)^2\ dx = (\int_a^b f(x)\ dx)^2$	
$\int_a^a f(x)\ dx = 0$	
$\int_a^b f(x)\ dx = \int_b^a f(x)\ dx$	

Solution:

Integral	Fill in 1-4	Finite?
$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\ dx$	4	*
$\int_{-1}^1 \frac{1}{x}\ dx$	2	
$\int_{-1}^1 \frac{1}{1+x^2}\ dx$	3	*
$\int_{-1}^1 \log\lvert x\rvert\ dx$	1	*

Identity	Check if true
$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$	*
$\int_a^b f(x)\ dx + \int_a^b g(x)\ dx = \int_a^b f(x) + g(x)\ dx$	*
$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$	*
$\int_a^b f(x)^2\ dx = (\int_a^b f(x)\ dx)^2$	
$\int_a^a f(x)\ dx = 0$	*
$\int_a^b f(x)\ dx = \int_b^a f(x)\ dx$	

Problem 3) (10 points)

Fill in the missing part into the empty box to make a true statement:

a) (2 points)

$\frac{d}{dx}\int_0^x f(t)\ dt =$ [ ] by the **fundamental theorem of calculus**.

b) (2 points)

$\int_0^x f(t)\ dt =$ ______ by the **fundamental theorem of calculus**.

c) (2 points)

The **mean value theorem** tells there is exists $a < x < b$ with $\frac{f(b)-f(a)}{b-a} =$ ______

d) (2 points)

A **probability distribution** satisfies $\int_{-\infty}^{\infty} f(x)\ dx = 1$ and ______ for all x.

e) (2 points)

For an improper integral $\int_a^b f(x)\ dx$, either $a = \infty$ or $b = \infty$ or f is ______ on $[a, b]$.

Solution:
a) $f(x)$
b) $F(x) - F(0)$ if F is the anti-derivative.
c) $f'(c)$ for some c.
d) $f \geq 0$
e) unbounded or discontinuous

Problem 4) Area computation (10 points)

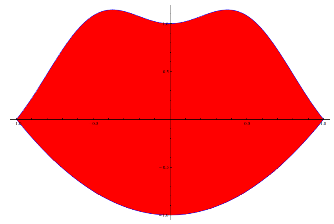
The region enclosed by the graphs of

$$f(x) = x^2 - 1$$

and

$$g(x) = 1 - x^2 + (1 - \cos(2\pi x))/6$$

models of the lips of **Rihanna**. Find the area.

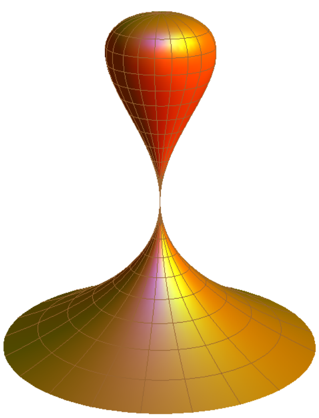


Solution:
The function g is above. The two graphs intersect at -1 and 1. We have

$$\int_{-1}^{1} 1 - x^2 + (1 - \cos(2\pi x))/6 - (x^2 - 1)\; dx = 2x - 2x^3/3 + 2x/6 - \sin(2\pi x)/(12\pi)|_{-1}^{1} = 3 \; .$$

The answer is $\boxed{3}$.

Problem 5) Volume computation (10 points)



The **kiss** is a solid of revolution for which the radius at height z is

$$z^2\sqrt{1-z}$$

and where $-1 \leq z \leq 1$. What is the volume of this solid? The name "kiss" is the official name for this quartic surface. Indeed, the top part has the shape of a **Hershey Kiss**. P.S. Creative **"exam product placement"** like this has been invented and patented by Oliver himself ...

Solution:
The area is $z^4(1-z)\pi$. We integrate this from -1 to 1 we first expand and write $\pi \int_{-1}^{1} z^4 - z^5 \; dz = 2\pi/5$. P.S. Oliver messed with you in the PS part. But Oliver (who writes this), always lies.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 \sqrt{1+x}\, dx$.

b) (2 points) $\int_0^1 \frac{1}{1+x^2}\, dx$

c) (2 points) $\int_2^e \frac{5}{3+x}\, dx$

d) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$.

e) (2 points) $\int_0^1 (x+1)^{10}\, dx$

Solution:
a) $(2/3)(\sqrt{8}-1)$
b) $\arctan(1) - \arctan(0) = \pi/4$
c) $5\log(3+e) - 5\log(5)$.
d) $-2\sqrt{x-1}|_0^1 = 2$
e) $(2^{11}-1)/11 = 2047/11$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives:

a) (2 points) $\int e^{23x} - x^{23}\, dx$

b) (2 points) $\int \frac{2}{x+3} + x^{1/23}\, dx$

c) (2 points) $\int \frac{23}{1+x} + 23\tan(x)\, dx$

d) (2 points) $\int \frac{1}{\sin^2(x)} + \frac{1}{x^3}\, dx$

e) (2 points) $\int \cos^2(3x)\, dx$

Jim Carrey in the movie "The number 23"

Solution:
a) $e^{23x}/23 - x^{24}/24 + c$.
b) $2\log(x+3) + (23/24)x^{24/23} + c$.
c) $23\log(1+x) - 23\log(\cos(x)) + c$.
d) $-\cot(x) - 1/(2x^2) + c$.
e) $\sin(6x)/12 + x/2 + c$.

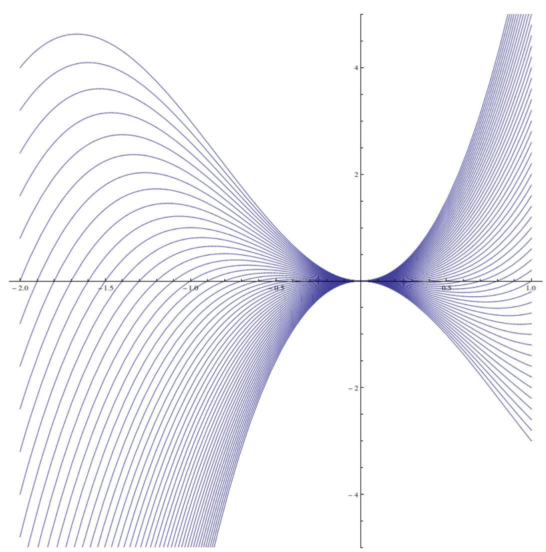
Problem 8) PDF's and CDF's (10 points)

Verify that the function $f(x) = \exp(-|x|)/2$ is a PDF.

Solution:
Yes, it is positive everywhere and $\int_0^\infty e^{-|x|}\ dx = 1$ and $\int_{-\infty}^0 e^{-|x|}\ dx = 1$ so that $\int_{-\infty}^\infty f(x)\ dx = 1$.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where c is a parameter.



a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which c is 0 a minimum?

e) (2 points) For which c does the catastrophe occur?

Solution:
a) $f'(x) = 6x^2 + 2cx = 2x(3x + c) = 0$ means either $x = 0$ or $x = -1$.
b) $f'(x) = 6x^2 - 6x = 0$ means either $x = 0$ or $x = 1$.
c) $f'(x) = 5x^2 - 2cx$ is always 0 independent of c.
d) $f''(0) = 2c = 2c$. We see that for $c > 0$ we have a minimum and for $c < 0$ a maximum.
e) At $c = 0$ the nature of the critical point changes. The parameter $c = 0$ is the catastrophe.