Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

4/8/2020: Second hourly

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. Handwrite your paper. It can be handwritten also electronically using an ipad but it needs to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ 20: False. Also, please sign your solutions.
- No books, calculators, computers, or other electronic aids are allowed. You can consult with a single page of your own handwritten notes, when writing the exam.
- The exam will be released at 9 AM on April 8. Try to do it during regular class time. You have to submit the paper within 24 hours by emailing it to knill@math.harvard.edu as an attachment. Submit it soon so that technical problems can be ironed out. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The method of substitution is based on the chain rule.

Solution:
Yes, that is how we have derived it.

2) T F $\int x^3 \, dx = 3x^2 + C$.

Solution:
Wrong way

3) T F The integral $\int_0^1 f(x) \, dx$ can be approximated by Riemann sums.

Solution:
We have defined it as such.

4) T F If $f(x) = 1$ everywhere, then $\int_a^b f(x) \, dx$ is the length of the interval $[a, b]$.

Solution:
Indeed the integral is $b - a$ then. Since we wrote $[a, b]$ this indicates that $a \leq b$.

5) T F If f is continuous, then $\int_a^b -f(x) \, dx = -\int_a^b f(x) \, dx$.

Solution:
Yes, we can take the $-$ sign outside the integral.

6) T F If f is a probability density function, then $f(b) - f(a)$ is the probability that the data are in the interval $[a, b]$.

7) T F One can find the anti-derivative of $\sin(4x)\cos(17x)$ using integration by parts.

8) T F The fundamental theorem of calculus implies $\int_a^b f'(x)\ dx = f(b) - f(a)$ if f' is a continuous function.

9) T F If $f(x) = 1/\sqrt{x}$, then the improper integral $\int_0^1 f(x)\ dx$ exists and gives a positive finite area.

Solution:
It is an improper integral for which the area does not exist.

10) T F The family $f_c(x) = x^2 + c$ experiences a catastrophe at $c = 0$.

Solution:
The nature of the critical point does not change when deforming c.

11) T F The anti derivative of $\log(5x)$ is $5x\log(5x) - 5x + C$.

Solution:
Differentiate the right hand side to check. It is almost right but not quite.

12) T F The volume of a cone of base radius 2 and height 2 is given by the integral $\int_0^2 \pi x^2\ dx$.

Solution:
Yes, the radius is x and the cross section area is πx^2.

13) T F The volume of a sphere can be derived from the volume of the cone and cylinder.

Solution:
Again, this is Archimedes insight.

14) T F If $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\ dx$ is finite.

Solution:
A counter example is $f(x) = 1/x$.

15) T **F** An integral is called improper if it is a convergent indefinite integral.

Solution:
These two terms are easy to mix up. Improper means that we either have a discontinuity of f or integrate over an infinite interval. Indefinite means that we do not specify bounds.

16) T **F** The anti-derivative of $\cot(x)$ is $1/(1+x^2)+C$.

Solution:
The derivative of $\arctan(x)$ is $1/(1+x^2)$, not tan itself.

17) T **F** Tic-Tac-Toe integration appears in the movie "Gifted".

Solution:
It is featured in the movie "stand and deliver".

18) T **F** Gabriel's trumpet is a solid with infinite volume.

Solution:
It has finite volume π.

19) T **F** The function $f(x) = e^x$ is called the exponential distribution.

Solution:
It is e^{-x} not e^x.

20) T **F** A PDF is the anti-derivative of the CDF.

Solution:
Wrong way.

Problem 2) Theorems (10 points) No justifications needed.

Fill in the missing part into the empty box to make a true statement.

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a) $\frac{d}{dx} \int_1^x f(t)\, dt =$ [ ] by the **fundamental theorem of calculus**.

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b) $\int_1^x f'(t)\, dt =$ [ ] by the **fundamental theorem of calculus**.

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c) A **probability distribution** is a piece-wise continuous function which is non-negative and satisfies the property [ ].

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d) The **improper integral** $\int_1^\infty \frac{1}{x^p}\, dx$ converges, if p satisfies the property [ ].

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e) Assume $f_c(x)$ is a **family of functions** such that for $c < 0$, there is no minimum and for $c > 0$ there is one minimum, then c is called a $\boxed{}$.

Solution:

a) $f(x)$.

b) $f(x) - f(1)$.

c) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

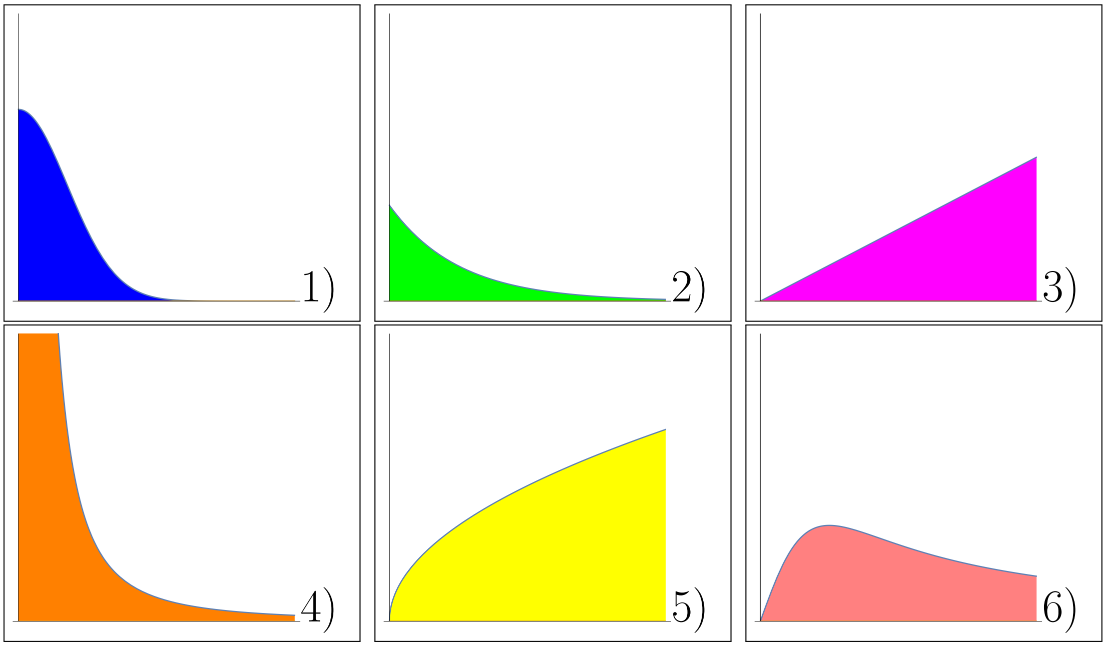
d) $p > 1$.

e) catastrophe.

Problem 3) Matching problem (10 points)

Match the following integrals with parts of the regions and indicate whether the integral represents a finite area.

Integral	Fill in 1-6	Convergent?
$\int_0^\infty x/2 \, dx$		
$\int_0^\infty \frac{1}{x^2} \, dx$		
$\int_0^\infty \sqrt{x} \, dx$		
$\int_0^\infty e^{-x} \, dx$		
$\int_0^\infty e^{-x^2} \, dx$		
$\int_0^\infty \frac{x}{1+x^2} \, dx$		



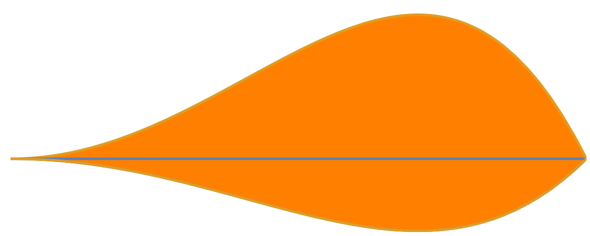
Solution:

3,4,5,2,1,6.

No, no, no, yes, yes, no.

Problem 4) Area computation (10 points)

Find the **area of the region** sandwiched between the graphs of $f(x) = x^4 - x^2$ and $g(x) = 2x^2 - 2x^4$. Document your work.



Solution:

The function $f(x) = x^4 - x^2$ is below. The graphs of the function intersect at $x = -1, 0, 1$. If one takes the region displayed in the figure one gets

$$\int_0^1 (2x^2 - 2x^4) - (x^4 - x^2)\, dx = 2/5$$

If one takes both regions, the answer is $4/5$. Both answers were counted correct.

Problem 5) Volume computation (10 points)

A solid has an elliptical cross section area of $A(x) = 1 - x^2$. It is located between $x = -1$ and $x = 1$. What is the **volume** V of that solid? Document your work.

Solution:

The volume is

$$\int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} (1 - x^2)\, dx = x - x^3/3|_{-1}^{1} = 4/3 \; .$$

Problem 6) Definite integrals (10 points, 2 points each)

Compute the following definite integrals. State the integration steps and names.

a) $\int_0^1 2x(1 + x^2)^{3/2}\, dx$

b) $\int_0^1 2x/(1 + x^2)\, dx$

c) $\int_0^1 x^3/(1 + x^2)\, dx$

d) $\int_0^1 \log(1+x)\; dx$

e) $\int_0^1 5xe^x\; dx$

Solution:
a) Use substitution $u = 1 + x^2, du = 2dx$. The answer is $\boxed{2(2^{5/2} - 1)/5}$.
b) Use substitution $u = 1 + x^2, du = 2xdx$ again. The answer is $\boxed{\log(2)}$.
c) Use substitution $u = 1 + x^2, x^2 = u - 1, du = 2xdx$ again and get $\boxed{(1 - \log(2))/2}$.
d) Use substitution $u = 1 + x$ to get $\int_1^2 \log(u)du$, then use integration by parts to get $u\log(u) - u|_1^2 = \boxed{2\log(2) - 1}$.
e) Use integration by parts to get $F(x)_0^1 = 5e^x(x-1)_0^1$. Evaluation gives $\boxed{5}$.

Problem 7) Anti derivatives (10 points, 2 points each)

Solve the indefinite integrals. State the integration method in each case.

a) $\int x^2 \log(x)\; dx$

b) $\int \cos^2(x)\sin(x)\; dx$

c) $\int \cos^2(2x)\; dx$

d) $\int x^2 \sin(x)\, dx$

e) $\int 3x^2/(1 + x^6)\, dx$

Solution:
a) Use integration by parts differentiating $\log(x)$ which gives $x^3 \log(x)/3 - \int_x^2 /3\, dx = x^3 \log(x)/3 - x^3/9 + C$.
b) Use substitution $u = \cos(x)$ to get $\cos^3(x)/3 + C$.
c) Use the double angle formula to write $\int (1 + \cos(4x))/2\, dx = x/2 + \sin(4)/8 + C$.
d) Use integration by parts twice or Tic-Tac-Toe to get $-x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C$.
e) Substitute $u = x^3$ to get $\arctan(x^3) + C$.

Problem 8) PDF's and CDF's (10 points)

We look at the function $f(x)$ which is $|x|$ on the interval $[-1, 1]$ and $f(x) = 0$ else.

a) (4 points) Verify that $f(x)$ is a PDF.

b) (3 points) If data are distributed according to $f(x)$, what is the probability of the data to be in the interval $[a, b] = [0, 1]$?

c) (3 points) What is the mean of the distribution f?

Solution:
a) The function is piecewise continuous, non-negative and satisfies $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{1} |x|/2\, dx = 2\int_0^1 x/2 = 1$.
b) The probability is $\int_0^1 f(x)\, dx = \int_0^1 x/2\, dx = 1/2$.
c) The mean is $\int_{-1}^{1} x|x|/2\, dx = \int_{-1}^{0} -x^2/2 + \int_0^1 x^2/2\, dx = 0$.

Problem 9) Catastrophes (10 points)

Let $f_c(x) = cx^2 + x^3$.

a) (3 points) Verify that $x = 0$ is a critical point for all c.

b) (4 points) Determine the stability of the critical point depending on c.

c) (3 points) For which value of c does a catastrophe appear?

Solution:
a) To find critical points, we differentiate. The derivative is $f_c(x) = 2cx+3x^2$. Independent of c, there is always the critical point $x = 0$.
b) The stability is determined by the second derivative $f_c''(x) = 2c + 6x$ which is $2c$ at $x = 0$. For $c > 0$ this is a minimum, for $c < 0$ this is a maximum. c) At $c = 0$ the number of minima changes. This is a catastrophe.