Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

4/8/2020: First hourly Practice A

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. Hand-write your paper. It can be handwritten also electronically using an ipad but it needs to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ 20: False. Also, please sign your solutions.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper.
- The exam will be released at 9 AM on April 8. Try to do it during regular class time. You have to submit the paper within 24 hours by emailing it to knill@math.harvard.edu as an attachment. Submit it soon so that technical problems can be ironed out. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F For any continuous function f we have $\int_0^1 3f(t)\,dt = 3\int_0^1 f(t)\,dt$.

2) T F For any continuous function $\int_0^3 f(t)\,dt = 3\int_0^1 f(t)\,dt$.

3) T F For any continuous function $\int_0^1 1 - f(t)\,dt = 1 - (\int_0^1 f(t)\,dt)$.

4) T F The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

5) T F The fundamental theorem of calculus implies that $\int_1^3 f'(x)\,dx = f(3) - f(1)$.

6) T F The integral $\pi\int_0^1 x^2\,dx$ gives the volume of a cone of height 1.

7) T F The anti-derivative of $1/\cos^2(x)$ is $\tan(x)$.

8) T F The function $F(x) = \int_0^x \tan(t^2)\,dt$ has the derivative $\tan(x^2)$.

9) T F The function $f(x) = \sin(x)/2$ on $[0,\pi]$ and $f(x) = 0$ else is a PDF

10) T F The identity $\frac{d}{dx}\int_1^2 \log(x)\,dx = \log(2) - \log(1)$ holds.

11) T F The function $f(x) = e^{-x^2/2}$ is called the normal distribution.

12) T F If $f < 1$, then $\int_0^2 f(x)\,dx$ can be bigger than 1.

13) T F An improper integral is an improperly defined definite indefinite integral.

14) T F The anti derivative $F(x)$ of $f(x)$ satisfies $F'(x) = f(x)$.

15) T F A parameter value c for which the number of minima are different for parameters smaller or larger than c is called a catastrophe.

16) T F If f is unbounded at 0, then $\int_0^1 f(x)\,dx$ is infinite.

17) T F If $f(-1) = 0$ and $f(1) = 1$ then $f' = 2$ somewhere on $(-1,1)$.

18) T F The anti-derivative of $\log(x)$ is $x\log(x) - x + C$, where log is the natural log.

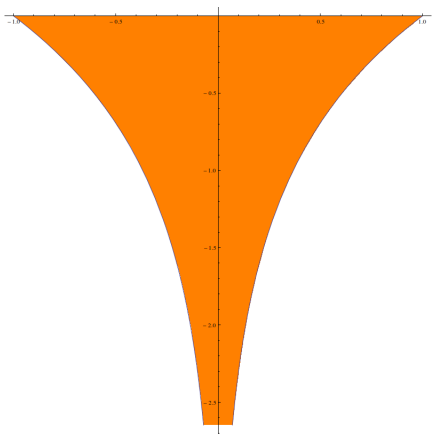
19) T F The sum $\frac{1}{n}[(\frac{0}{n})^2 + (\frac{1}{n})^2 + ... + (\frac{n-1}{n})^2]$ converges to $1/3$ in the limit $n \to \infty$.

20) T F The **improper integral** $\int_1^\infty \frac{1}{x^2}\,dx$ represents a finite area.

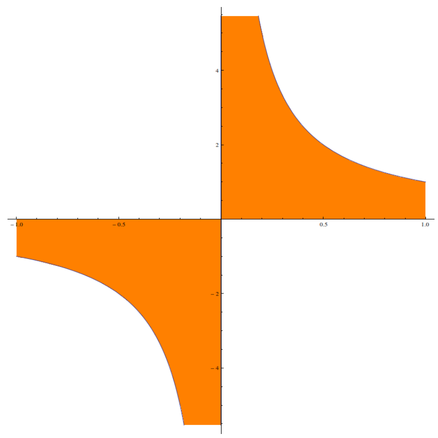
Problem 2) Matching problem (10 points) No justifications are needed.

a) (4 points) Match the following integrals with the regions and indicate whether the integral represents a finite area.

Integral	Fill in 1-4	Finite?
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\ dx$		
$\int_{-1}^{1} \frac{1}{x}\ dx$		
$\int_{-1}^{1} \frac{1}{1+x^2}\ dx$		
$\int_{-1}^{1} \log\|x\|\ dx$		



1)



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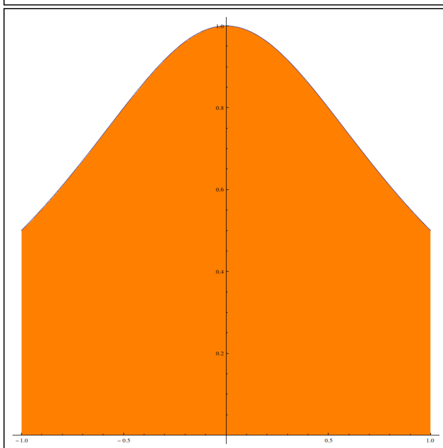

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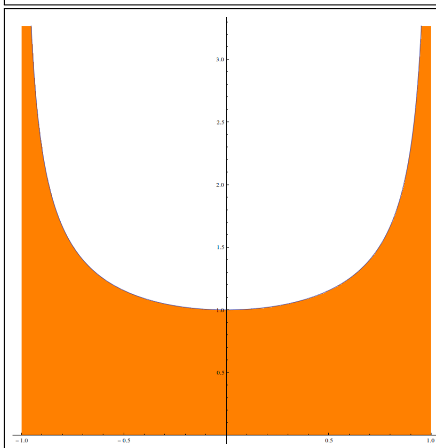


4)

b) (6 points) Which of the following properties are always true. This means which are true for all choices of continuous functions and all choices of a, b, c.

Identity	Check if true
$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x) + g(x)\,dx$	
$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$	
$\int_a^b f(x)^2\,dx = (\int_a^b f(x)\,dx)^2$	
$\int_a^a f(x)\,dx = 0$	
$\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$	

Problem 3) (10 points)

Fill in the missing part into the empty box to make a true statement:

a) (2 points)

$\frac{d}{dx}\int_0^x f(t)\,dt =$ [ ] by the **fundamental theorem of calculus**.

b) (2 points)

$\int_0^x f(t)\,dt =$ [ ] by the **fundamental theorem of calculus**.

c) (2 points)

The **mean value theorem** tells there is exists $a < x < b$ with $\frac{f(b)-f(a)}{b-a} =$ [ ]

d) (2 points)

A **probability distribution** satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and [ ] for all x.

e) (2 points)

For an improper integral $\int_a^b f(x)\,dx$, either $a = \infty$ or $b = \infty$ or f is [ ] on $[a, b]$.

Problem 4) Area computation (10 points)

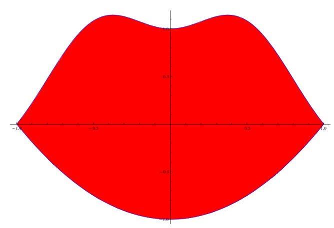
The region enclosed by the graphs of

$$f(x) = x^2 - 1$$

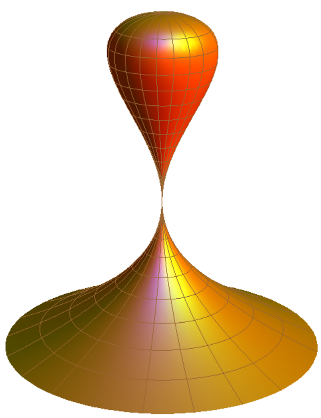
and

$$g(x) = 1 - x^2 + (1 - \cos(2\pi x))/6$$

models of the lips of **Rihanna**. Find the area.



Problem 5) Volume computation (10 points)



The **kiss** is a solid of revolution for which the radius at height z is

$$z^2\sqrt{1-z}$$

and where $-1 \leq z \leq 1$. What is the volume of this solid? The name "kiss" is the official name for this quartic surface. Indeed, the top part has the shape of a **Hershey Kiss**. P.S. Creative **"exam product placement"** like this has been invented and patented by Oliver himself ...

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 \sqrt{1+x}\,dx$.

b) (2 points) $\int_0^1 \frac{1}{1+x^2}\,dx$

c) (2 points) $\int_2^e \frac{5}{3+x}\, dx$

d) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$.

e) (2 points) $\int_0^1 (x+1)^{10}\, dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives:

a) (2 points) $\int e^{23x} - x^{23}\, dx$

b) (2 points) $\int \frac{2}{x+3} + x^{1/23}\, dx$

c) (2 points) $\int \frac{23}{1+x} + 23\tan(x)\, dx$

d) (2 points) $\int \frac{1}{\sin^2(x)} + \frac{1}{x^3}\, dx$

e) (2 points) $\int \cos^2(3x)\, dx$

Jim Carrey in the movie "The number 23"

Problem 8) PDF's and CDF's (10 points)

Verify that the function $f(x) = \exp(-|x|)/2$ is a PDF.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where c is a parameter.

a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which c is 0 a minimum?

e) (2 points) For which c does the catastrophe occur?