Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

3/4/2020: First hourly Practice D

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 75 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

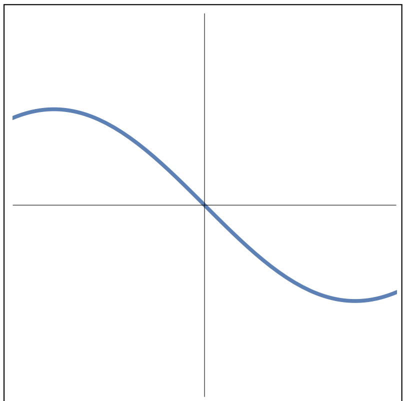
1)	T	F	The function arcsin(x) is defined as $1/\sin(x)$.
2)	T	F	The function $f(x) = \sin(1/x^2)$ can be defined at 0 so that it becomes a continuous everywhere on the real line.
3)	T	F	The function $x/\sin(x)$ can be defined at $x = 0$ so that it becomes a continuous function on the real line.
4)	T	F	The function $f(x) = \sin^2(x)/x^2$ has the limit 1 at $x = 0$.
5)	T	F	The function $f(x) = 1/\log\|x\|$ has the limit 1 at $x = 0$.
6)	T	F	The function $f(x) = (1+h)^{x/h}$ has the property that $Df(x) = [f(x+h) - f(x)]/h = f(x)$.
7)	T	F	$\cos(3\pi/2) = 1$.
8)	T	F	If a function f is continuous on the interval $[3, 10]$, then it has a global maximum on this interval.
9)	T	F	The reciprocal rule assures that $d/dx(1/g(x)) = 1/g(x)^2$.
10)	T	F	If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x \to 0}(f(x)/g(x)) = 1$
11)	T	F	An inflection point is a point where the function $f''(x)$ changes sign.
12)	T	F	If $f''(x) > 0$ then f is concave up at x.
13)	T	F	The chain rule assures that $d/dx f(g(x)) = f'(x)g'(x)$.
14)	T	F	The function $f(x) = 1/x + \log(x)$ is continuous on the interval $[1, 2]$.
15)	T	F	If we perform the Newton step for the function $\exp(x)$, we get the map $T(x) = x - 1$.
16)	T	F	The graph of the function $f(x) = x/(1+x^2)$ has slope 1 at 0.
17)	T	F	There is a differentiable function for which $f'(0) = 0$ but for which 0 is not a local extremum.
18)	T	F	The second derivative test assures that $x = p$ is a local minimum if $f'(p) = 0$ and $f''(p) < 0$.
19)	T	F	The identity $(x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ holds for all $x \neq 1$.
20)	T	F	The slope of the tangent at a point $(x, f(x))$ of the graph of a differentiable function f is equal to $1/f'(x)$.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs. Naturally, only 10 of the 12 graphs will appear.

Function	Enter 1-12
$\cot(x)$	
$\cos(2x)$	
$2x$	
$\tan(x)$	
$\log(1/\|x\|)$	

Function	Enter 1-12
x^2	
$\exp(x)$	
$-\sin(x)$	
x^3	
$\mathrm{sinc}(x)$	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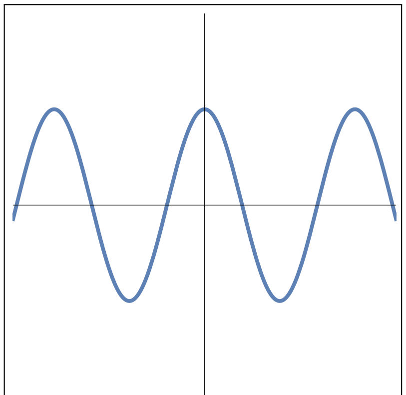
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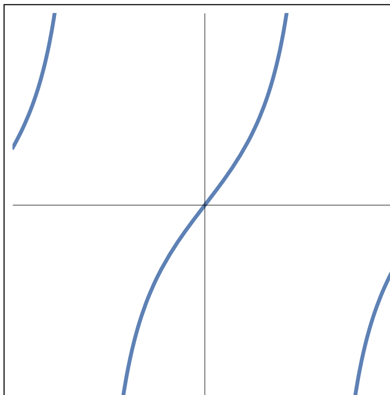
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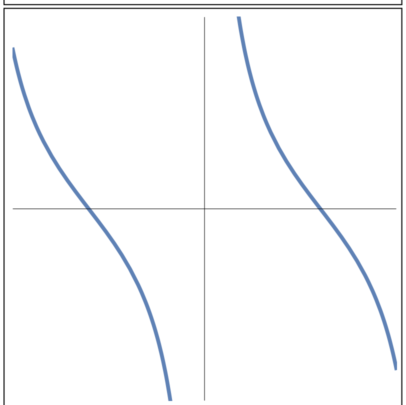
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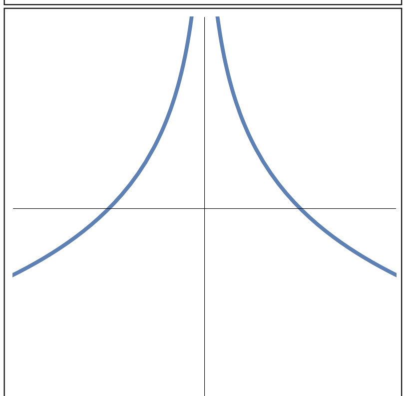
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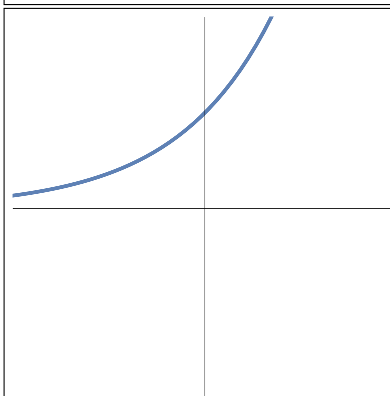
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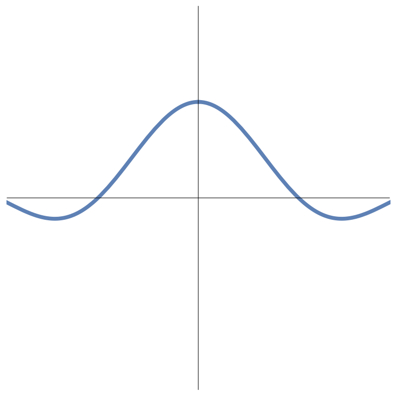

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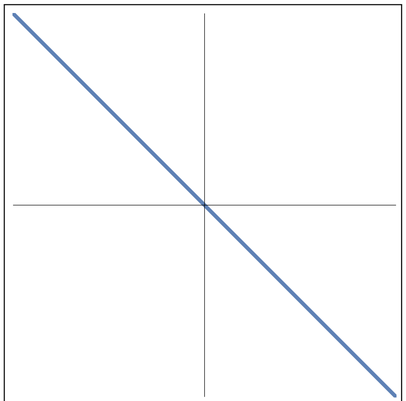

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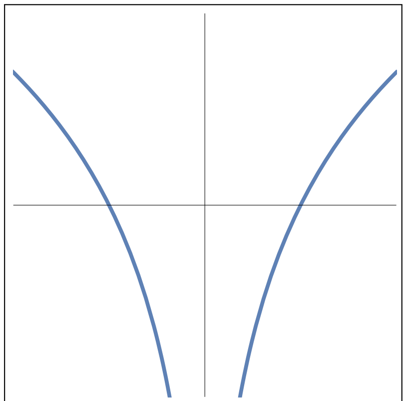

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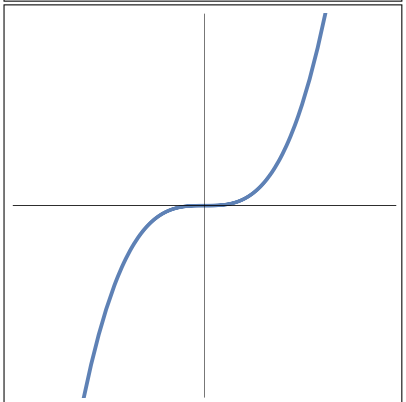

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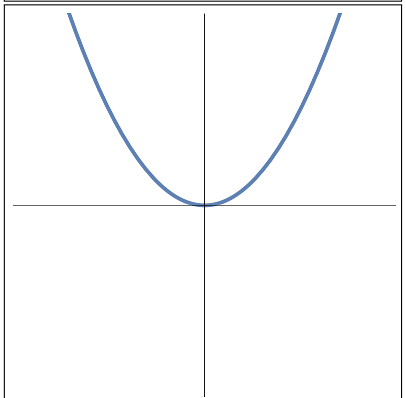

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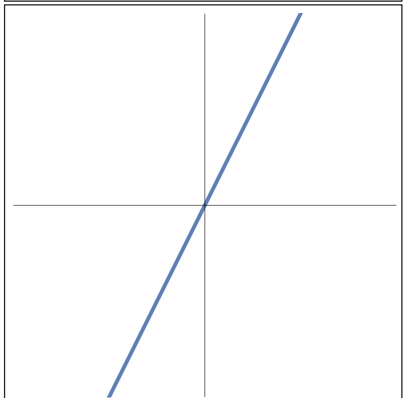
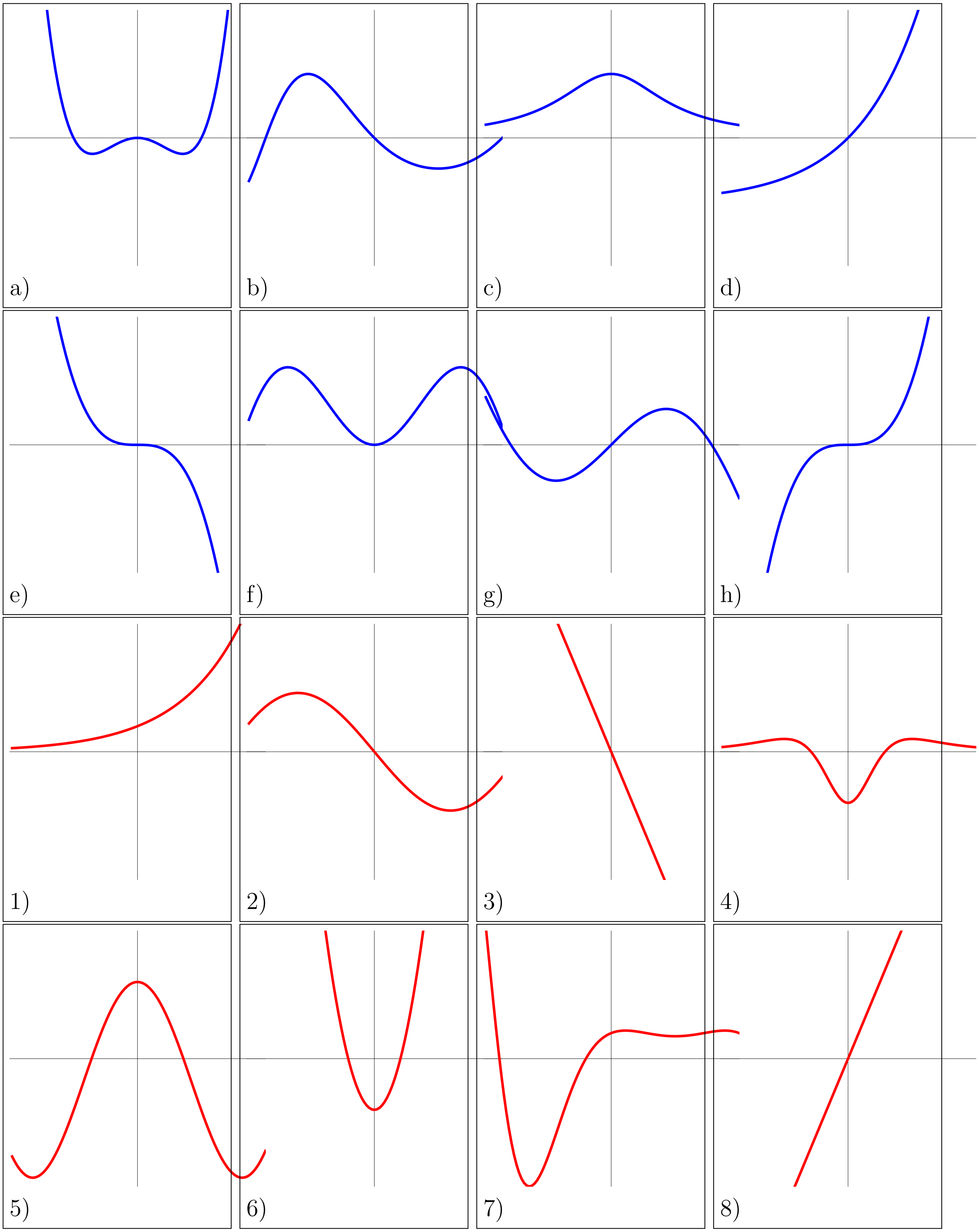


12)

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).

Function	Second derivative (Enter 1- 8 here)
a)	
b)	
c)	
d)	
e)	
f)	
g)	
h)	



Problem 4) Continuity (10 points)

Some of the following functions might a priori not be defined yet at the point a. In each case, decide whether f can be made a continuous function by assigning a value $f(a)$ at the point a. If no such value exist, state that the function is not continuous.

a) (2 points) $f(x) = \frac{(x^3-1)}{(x-1)}$, at $x = 1$

b) (2 points) $f(x) = \sin(\frac{1}{x}) + \cos(x)$, at $x = 0$

c) (2 points) $f(x) = \sin(\frac{1}{\log(|x|)})$, at $x = 0$

d) (2 points) $f(x) = \log(|\sin(x)|)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Problem 5) Chain rule (10 points)

a) (2 points) Write $1+\cot^2(x)$ as an expression which only involves the function $\sin(x)$.

b) (3 points) Find the derivative of the function $\text{arccot}(x)$ by using the chain rule for

$$\cot(\text{arccot}(x)) = x \ .$$

c) (2 points) Write $1+\tan^2(x)$ as an expression which only involves the function $\cos(x)$.

d) (3 points) Find the derivative of the function $\arctan(x)$ by using the chain rule for

$$\tan(\arctan(x)) = x \ .$$

Remark: even if you should know the derivatives of arccot or arctan, we want to see the derivations in b) and d).

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{\cos(3x)}{\cos(x)}$

b) (2 points) $f(x) = \sin^2(x)\log(1 + x^2)$

c) (2 points) $f(x) = 5x^4 - \frac{1}{x^2+1}$

d) (2 points) $f(x) = \tan(x) + \exp(-\sin(x^2))$

e) (2 points) $f(x) = \frac{x^3}{(1+x^2)}$

Problem 7) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions f at $x = 0$ or state (providing reasoning as usual) that the limit does not exist.

a) (2 points) $f(x) = \frac{\sin(3x)}{\sin(x)}$

b) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

c) (2 points) $f(x) = \sin(\log(|x|))$

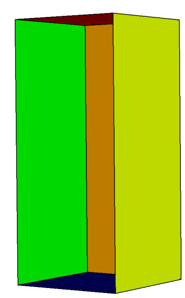
d) (2 points) $f(x) = \tan(x)\log(x)$

e) (2 points) $f(x) = \frac{(5x^4-1)}{(x^2+1)}$

Problem 8) Extrema (10 points)

A rectangular shoe-box of width x, length x and height y is of volume 2 so that $x^2y = 2$. The surface area adds up three rectangular parts of size $(x \times y)$ and 2 square parts of size $(x \times x)$ and leads to

$$f = 2x^2 + 3xy .$$



a) (2 points) Write down the function $f(x)$ of the single variable x you want to minimize.

b) (6 points) Find the value of x for which the surface area is minimal.

c) (2 points) Check with the second derivative test, whether the point you found is a local minimum.

Problem 9) Global extrema (10 points)

In this problem we study the function $f(x) = 3x^5 - 5x^3$ on the interval $[-2, 2]$.

a) (2 points) Find all roots of f.
b) (2 points) Find all local extrema of the function.
c) (2 points) Use the second derivative test to analyze the critical points, where applicable.
d) (2 points) Find the **global** maximum and minimum of f on the interval $[-2, 2]$. e) (2 points) Perform one Newton step for the function $f(x) = 3x^5 - 5x^3$ starting at the

point $x = 2$.