Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

3/4/2020: First hourly Practice B

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 75 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If f is concave up on $[0,1]$ and concave down on $[1,2]$ then 1 is an inflection points.

2) T F The function $f(x) = \exp(x)$ has the root $x = 1$.

3) T F $\log(\exp(1)) = 1$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

4) T F The chain rule assures that $d/dx f(f(x)) = f'(f(x))f'(x)$.

5) T F The function $x^2/(1+x^2)$ is continuous everywhere on the real axes.

6) T F The function $\cot(x)$ is the inverse of the function $\tan(x)$.

7) T F The Newton method is $T(x) = x + f(x)/f'(x)$.

8) T F $\cos(\pi/2) = 1/2$.

9) T F If a function f is differentiable on $[-1,1]$, then there is a point x in that interval where $f'(x) = 0$.

10) T F The chain rule assures that $d/dx(g(x^2)) = 2xg'(x^2)$.

11) T F We have $\lim_{x\to\infty}((x^2+1)/x^2) = 1$

12) T F An inflection point is a point, where the function $f'(x)$ changes sign.

13) T F If $f''(-2) > 0$ then f is concave up at $x = -2$.

14) T F The intermediate value theorem assures that the continuous function $x + \sin(x) = 0$ has a root.

15) T F We can find a value b and define $f(0) = b$ such that the function $f(x) = (x^{28}-1)/(x^2-1)$ is continuous everywhere.

16) T F If the third derivative $f'''(x)$ is negative and $f''(x) = 0$ then f has a local maximum at x.

17) T F If $f(x) = x^2$ then $Df(x) = f(x+1) - f(x)$ has a graph which is a line.

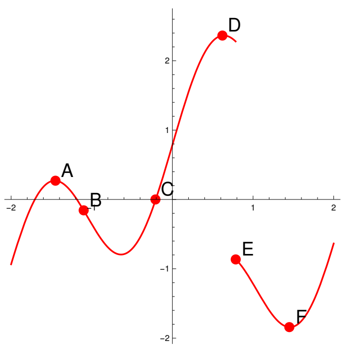
18) T F The quotient rule is $d/dx(f/g) = f'(x)/g'(x)$.

19) T F With $Df(x) = f(x+1) - f(x)$, we have $D(1+a)^x = a(1+a)^x$.

20) T F It is true that $\log(5)e^x = e^{x\log(5)}$ if $\log(x)$ is the natural log.

Problem 2) Matching problem (10 points) No justifications are needed.

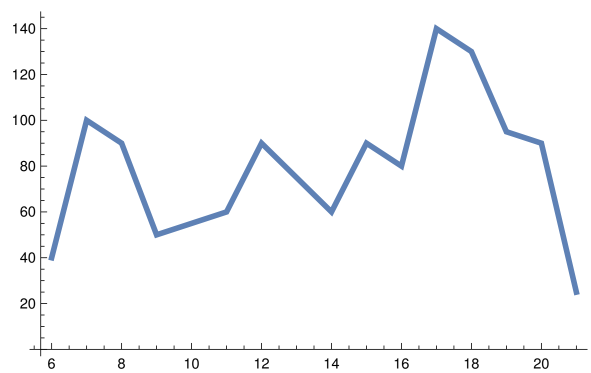
a) (6 points) You see the graph of a function $f(x)$ defined on $[-2, 3]$. Various points $(x, f(x))$ are marked. Match them:



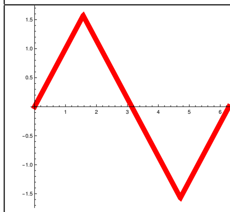
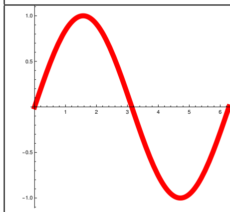
Point x is	Fill in A-F
Local maximum	
Root	
Inflection point	
Discontinuity	
Global maximum	
Local minimum	

b) (2 points) the **Harvard recreation** publishes regularly a graph of a function $f(x)$ which shows the number of people at the **Mac** gym as a function of time. At 5 o'clock, there are in average 120 visitors, at 9 in the morning, there are 60 people working out. By the intermediate value theorem, there must be a moment at which exactly $\pi^4 = 97.5...$ visitors are present. This is obviously nonsense. Where is the flaw?

Reason	Check one
No differentiability	
Statistical glitch	
No Continuity	
Inaccurate data	

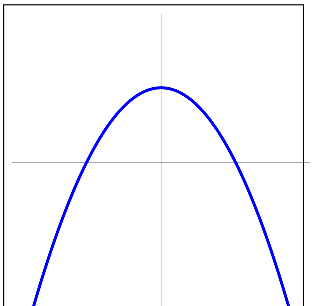


c) (2 points) In front of the **"Class of 1959 Chapel"** at the Harvard business school is an amazing clock: a marble tower contains a steel pole and a large bronze ball which moves up and down the pole indicating the time of the day. As the ball moves up and down the pole, lines with equal distance on the tower indicate the time. At noon, the sphere is at the highest point. At midnight it is at the bottom. It moves the same distance in each hour. If we plot the hight of the sphere as a function of time, which graph do we see?

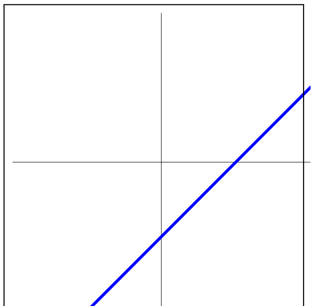
The height function	Check which applies
	
	

Problem 3) Matching problem (10 points) No justifications are needed.

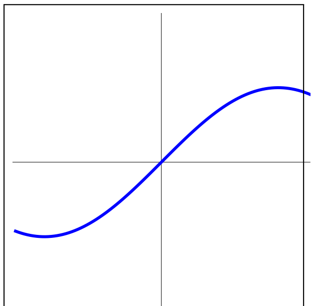
Match the graph of the functions f in $a) - h)$ with the derivatives f' in 1)-8).



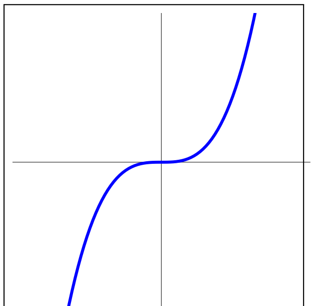
a) $\rightarrow$ ☐



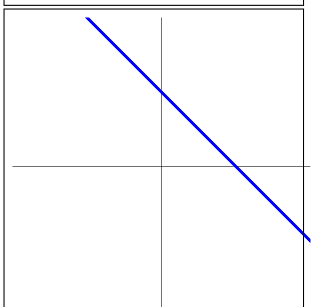
b) $\rightarrow$ ☐



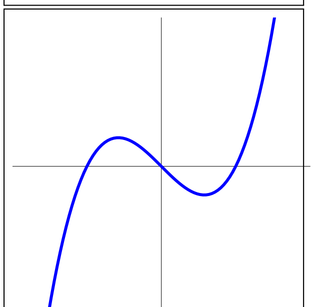
c) $\rightarrow$ ☐



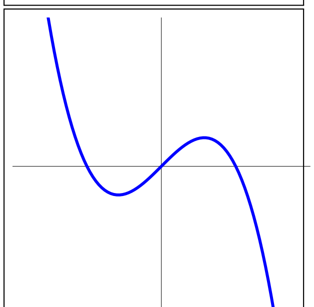
d) $\rightarrow$ ☐



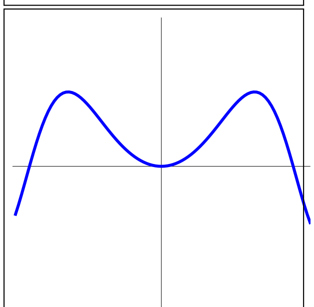
e) $\rightarrow$ ☐



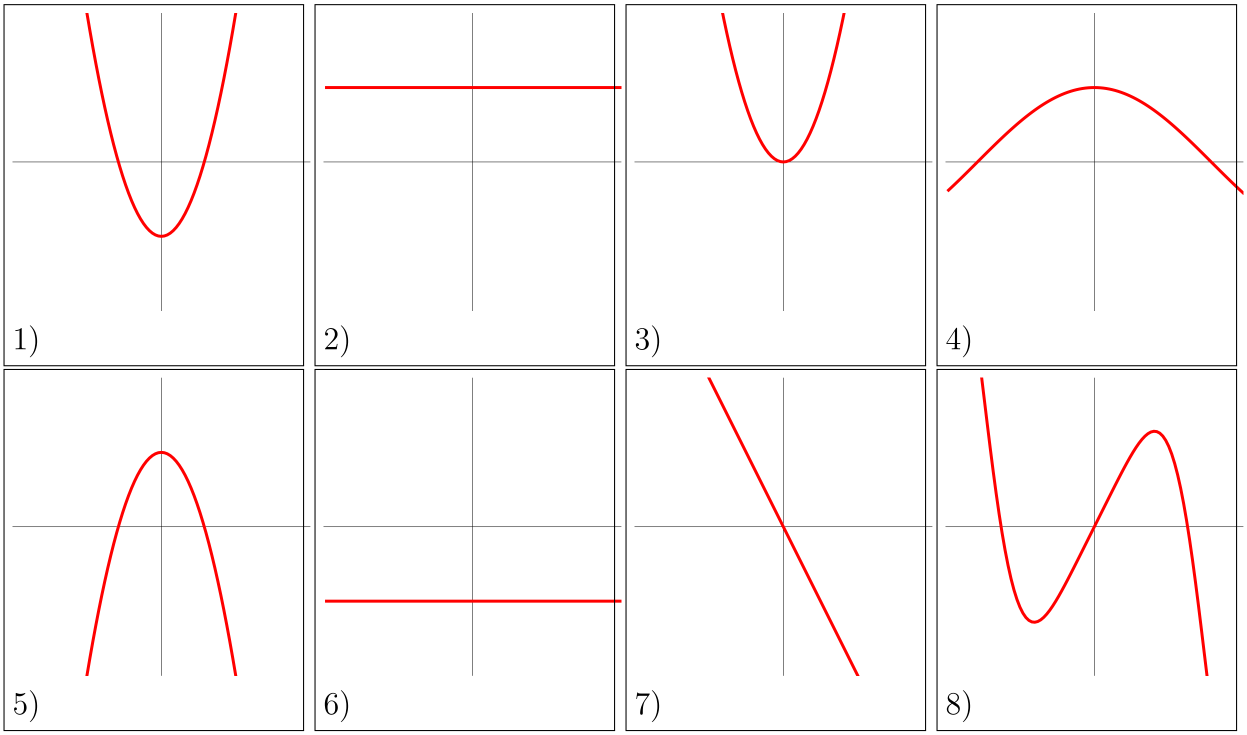
f) $\rightarrow$ ☐



g) $\rightarrow$ ☐



h) $\rightarrow$ ☐



Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x \to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{x^3-8}{x-2}$, at $x_0 = 2$

b) (2 points) $f(x) = \sin(\sin(\frac{1}{x})) - \tan(x)$, at $x_0 = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x_0 = 0$

d) (2 points) $f(x) = \frac{\exp(x)-1}{\exp(5x)-1}$, at $x_0 = 0$

e) (2 points) $f(x) = \frac{x-1}{x}$, at $x_0 = 0$

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sin(7x) + (1 + x^2)$.

b) (2 points) $f(x) = \frac{\sin(7x)}{(1+x^2)}$.

c) (2 points) $f(x) = \sin(7 + x^2)$.

d) (2 points) $f(x) = \sin(7x)(1 + x^2)$.

e) (2 points) $f(x) = \sin(7x)^{(1+x^2)}$

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions f at $x = 0$:

a) (2 points) $f(x) = \frac{(1-\exp(11x))}{(1-\exp(3x))}$

b) (2 points) $f(x) = \frac{\sin(\sin(5x))}{\sin(7x)}$

c) (2 points) $f(x) = \frac{\log(x)}{\log(5x)}$

d) (2 points) $f(x) = \frac{x^2 \cos(x)}{\sin^2(x)}$

e) (2 points) $f(x) = \frac{(1+1/x^2)}{(1-1/x^2)}$

Problem 7) Trig functions (10 points)

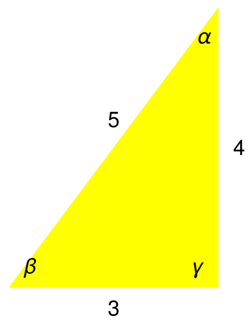
A triangle with side lengths $3, 4, 5$ has a right angle. Let $\alpha < \beta < \gamma$ denote the angles ordered by size.

a) (4 points) What are the numerical values of $\cos(\alpha), \cos(\beta), \cos(\gamma), \sin(\gamma)$?

b) (2 points) Find the numerical value of $\tan(\alpha)$ and $\cot(\alpha)$.

The next problem is independent of the previous two.

c) (4 points) Find the derivative of the inverse function of $\arcsin(x)$ by starting with the identity $x = \sin(\arcsin(x))$. Your derivation of $\arcsin'(x)$ should convince somebody who does not know the identity already.



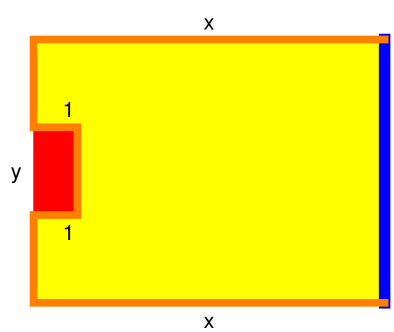
Problem 8) Extrema (10 points)

A tennis field of width x and length y contains a fenced referee area of length 2 and width 1 within the field and an already built wall. The circumference a fence satisfies $2x + y + 2 = 10$, (an expression which still can be simplified). We want to maximize the area $xy - 2$.

a) (2 points) On which interval $[a, b]$ does the variable x make sense? Find a function $f(x)$ which needs to be maximized.

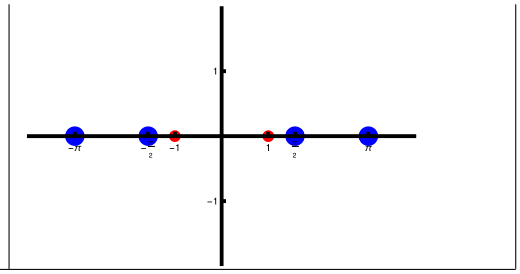
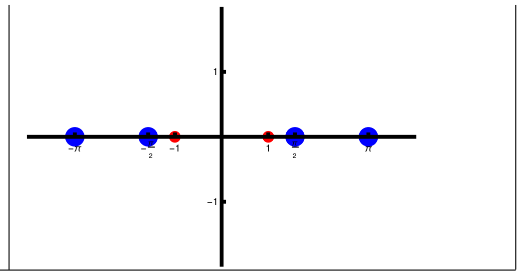
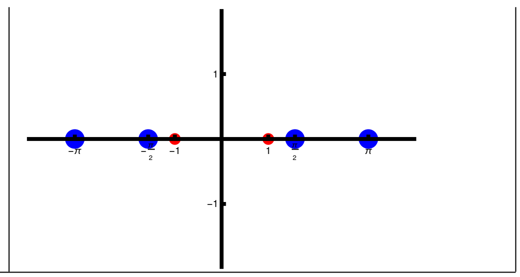
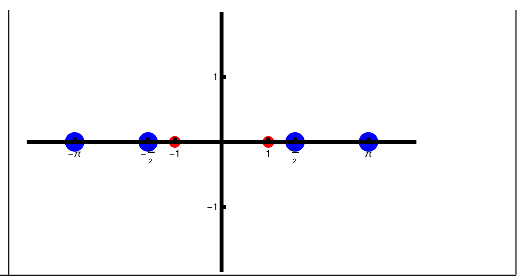
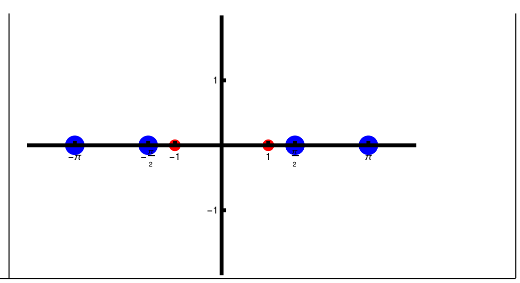
b) (6 points) Find the local maximum of x and check it with the second derivative test.

c) (2 points) What is the global maximum of f on $[a, b]$?



Problem 9) Trig functions (10 points)

a) In the following five problems, find the numerical value and then draw the graph of the function.

Problem	Answer	Graph
A) (2 points) What is $\sin(\pi/3)$? Plot $\sin(x)$.		
B) (2 points) What is $\cos(5\pi/2)$? Plot $\cos(x)$.		
C) (2 points) Find $\arctan(1)$ Plot $\arctan(x)$.		
D) (2 points) What is $\log(1)$ Plot $\log\|x\|$.		
E) (2 points) What is $\arcsin(\sqrt{3}/2)$. Plot $\arcsin(x)$		

b) Simplify the following terms. log denotes the natural log and $\log_{10}$ denotes the log to the base 10. All answers are integers.

A) (2 points) $\exp(\log(2))$

B) (2 points) $e^{\log(2)3}$
C) (2 points) $\log(\log(e))$
D) (2 points) $\exp(\log(2) + \log(3))$
E) (2 points) $\log_{10}(10000)$