Unit 2: Worksheet

In this lecture, we get acquainted with the most important functions.

Trigonometric functions

The cosine and sine functions can be defined geometrically by the coordinates $(\cos(x), \sin(x))$ of a point on the unit circle. The tangent function is defined as $\tan(x) = \sin(x)/\cos(x)$. Define also $\cot(x) = 1/\tan(x)$. [1]

$$\begin{aligned} \sin(x) = & \text{ opposite/hypothenuse (SOH)} \\ \cos(x) = & \text{ adjacent/hypothenuse (CAH)} \\ \tan(x) = & \text{ opposite/adjadcent (TOA)} \end{aligned}$$

a): Find $\cos(\pi/3)$, $\sin(\pi/3)$.

b): Find the roots of $f(x) = \cos(x)$ and $f(x) = \sin(x)$.

c): Find the roots of $f(x) = \cos(x-1)$ and $f(x) = \sin(x+1)$.

d): Find $\tan(3\pi/2)$ and $\cot(3\pi/2)$.

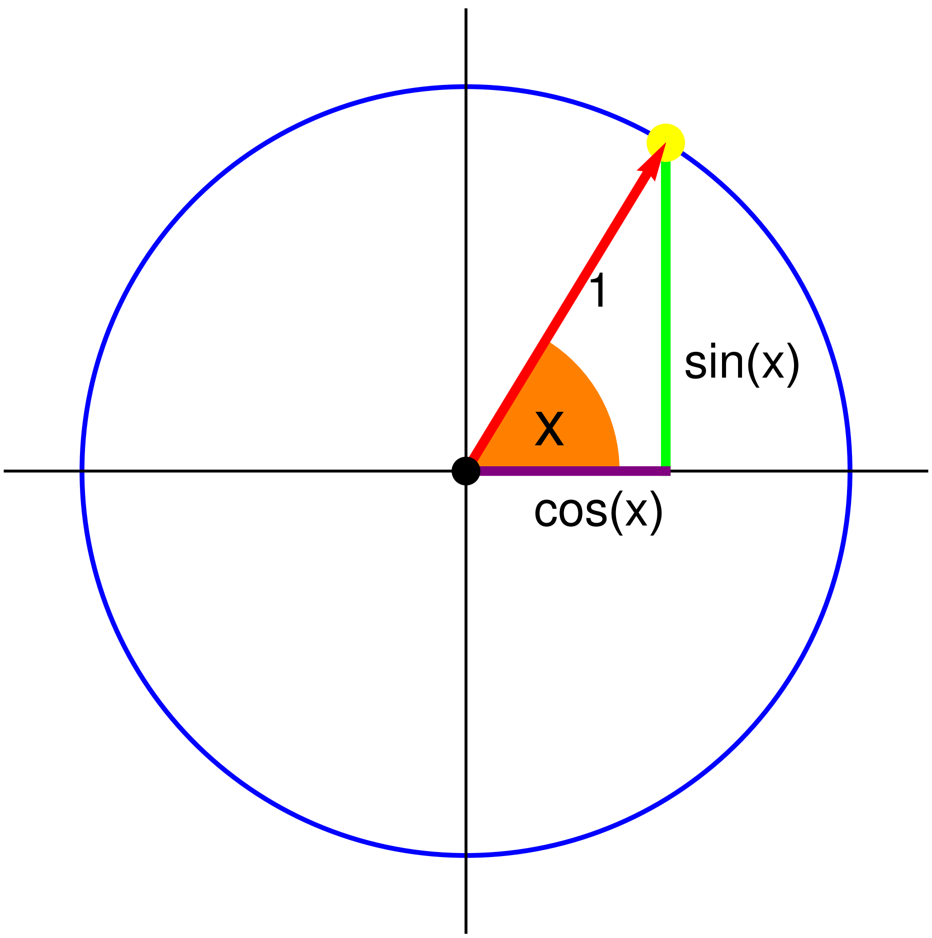
e): Find $\cos(3\pi/2)$ and $\sin(3\pi/2)$.

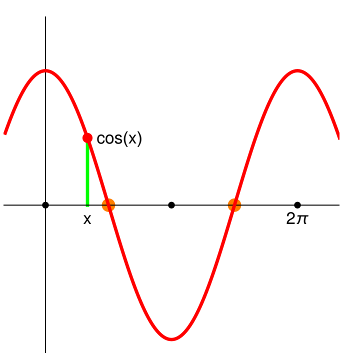
f): Find $\tan(\pi/4)$ and $\cot(\pi/4)$.

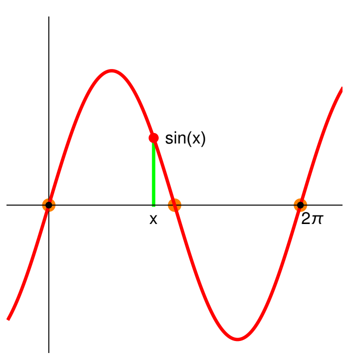
[1] We will hardly $\sec(x) = 1/\cos(x), \csc(x) = 1/\sin(x)$ as not much is gained by this abbreviation

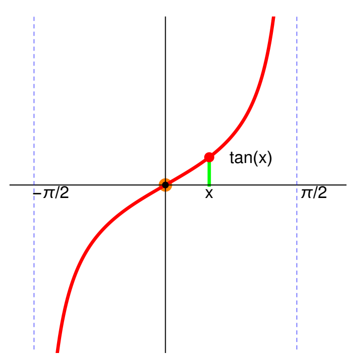
We have $\cos(\pi/2 - x) = \sin(x)$ and $\sin(\pi/2 - x) = \cos(x)$. **Pythagoras** gives:

$$\boxed{\cos^2(x) + \sin^2(x) = 1}$$



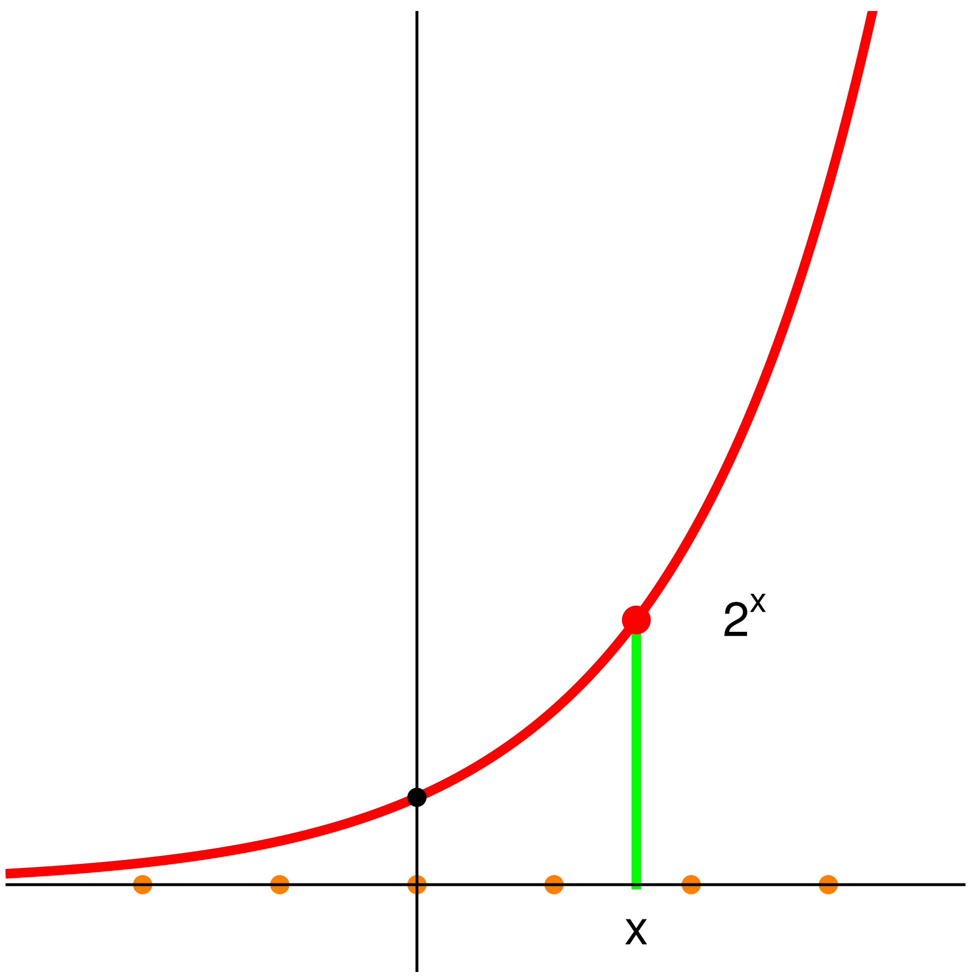






The exponential function

The function $f(x) = 2^x$ is first defined for positive integers like $2^{10} = 1024$, then for all integers with $f(0) = 1$, $f(-n) = 1/f(n)$. By taking roots, one can define $2^{3/2} = 8^{1/2} = \sqrt{8} = 2.828...$. Since the function 2^x is monotonone on the set of rationals, we can fill the gaps and define $f(x)$ for any real x. By taking square roots again and again for example, we see $2^{1/2}, 2^{1/4}, 2^{1/8}, ...$ we approach $2^0 = 1$.



a): What is 2^{-5}?
b): Find $2^{1/2}$.
c): Find $27^{1/3}$.
d): Find $64^{-1/4}$.
e): What is larger: $A = 2^{3/4}$ or $B = 2^{4/5}$?
f): What is larger: $A = 2^{-1/2}$ or $B = 2^{-1/3}$?

There is nothing special about 2 and we can take any positive base a and define the exponential a^x. It satisfies $a^0 = 1$ and the remarkable rule:

$$\boxed{a^{x+y} = a^x \cdot a^y}$$

It is spectacular because it provides a link between addition and multiplication.

We will especially consider the **exponential** $\boxed{\exp_h(x) = (1+h)^{x/h}}$, where h is a positive parameter. This super cool exponential satisfies $\exp_h(x+h) = (1+h)\exp_h(x)$ so that

$$[\exp_h(x+h) - \exp_h(x)]/h = \exp_h(x) .$$

We will see this relation again.
For $h = 1$, we have the function 2^x. In the limit $h \to 0$, we get the important exponential function $\exp(x)$ which we also call e^x.
For $x = 1$, we get the **Euler number** $e = e^1 = 2.71828\ldots$.

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