INTRODUCTION TO CALCULUS

MATH 1A

Lecture 36: Related rates

Before we continue with integration, we include a short flash-back on differentiation. which allows us to review the **chain rule**

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

a rule which will lead to the "substitution" integration technique. Since the chain rule is often perceived hard, it is good to review it before launching into integration techniques.
Related rates problem deal with a relation for variables. Differentiation gives a relation between the derivatives (rate of change). In all these problems, we have an **equation** and a **rate**. You can then solve for the rate which is asked for.
Example: Hydrophilic **water gel spheres** have volume $\boxed{V(r(t)) = 4\pi r(t)^3/3}$ and expand at a rate $\boxed{V' = 30}$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

Example: A **wine glass** has a shape $y = x^2$ and volume $\boxed{V(y) = y^2\pi/2}$. Assume we slurp the wine with constant rate $\boxed{V' = -0.1}$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.

Example: A **ladder** has length 1. Assume slips on the ground away with constant speed $\boxed{x' = 2}$. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$ if $\boxed{x^2(t) + y^2(t) = 1}$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

Example: A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume $\boxed{y' = -7}$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

Image source: http://www.dmfco.com

Example: A **canister of oil** releases oil so that the area grows at a constant rate $\boxed{A' = 5}$. With what rate does the radius increase? **Solution.** We have $\boxed{A(r) = r^2\pi}$ and so $5 = A'(r) = 2rr'\pi$. Solving for r' gives $r' = 5/(2r\pi)$.

Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time use the given $\boxed{\text{rate of change}}$ and solve for the unknown rate of change. Since related change problems are often difficult to parse. We have the $\boxed{\text{rule}}$ and given $\boxed{\text{rate of change}}$ boxed.

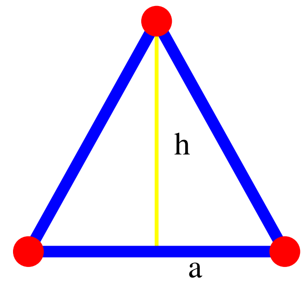
Homework

1 The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and $\boxed{V' = -5}$. Find the rate p' with which the pressure increases if $V = 10$ and $p = 5$.

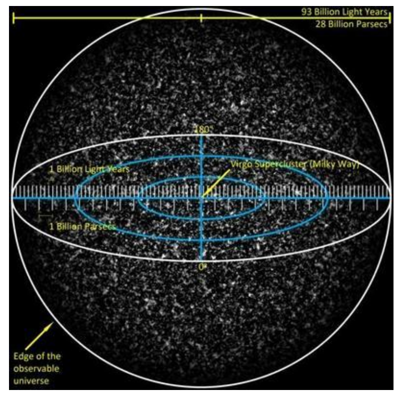
2 Assume the **total production rate** P of an Oculus Rift is constant $P = 100$ and given by the **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$. Assume labor is increased at a rate $\boxed{L' = 2}$. What is the cost change K'? Evaluate this at $K = 125$ and $L = 64$.

3 You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\boxed{\phi' = 5\pi/180 = \pi/36}$ radiants per second (which means 5 degrees per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$ to get ϕ for $x = 0$.

4 An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $\boxed{h' = -2}$. With what rate does a increase if $h = 1/2$?



5 There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $\boxed{V' = 100\pi^2 r^2}$. What is r'? Evaluate it for $r = 47$ (billion light years).



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