Unit 26: Review

Scope

The second midterm covers units 16-25: catastrophes, integrals, fundamental theorem of calculus and anti-derivatives, areas, volumes, improper integrals, probability distributions, substitution and integration by parts.

Main points

Important integration techniques: **use substitution**, (write down u, du, dx) **use integration by parts** ($\int udv = uv - \int vdu$).

Catastrophes are parameter values where the number of minima changes. To find the parameter, look where $f''(x_c)$ becomes zero at the critical point x_c.

Definite integrals $F(x) = \int_0^x f(t)\, dt$ were defined as a limit of Riemann sums.

A function $F(x)$ satisfying $F' = f$ is called the anti-derivative of f. The general anti-derivative is $F + c$ where c is a constant.

The **fundamental theorem of calculus** tells $d/dx \int_0^x f(x)\, dx = f(x)$ and $\int_0^x f'(x)\, dx = f(x) - f(0)$.

The integral $\int_a^b g(x) - f(x)\ dx$ is the **signed area between the graphs** of f and g. Places, where $f < g$ are counted negative. When area is asked, split things up.

The integral $\int_a^b A(x)\, dx$ is a **volume** if $A(x)$ is the area of a slice of the solid perpendicular to a point x on an axes.

Write **improper integrals** as limits of definite integrals $\int_1^\infty f(x)\,dx = \lim_{R\to\infty} \int_1^R f(x)\,dx$. We similarly treat points, where f is discontinuous.

One can use CDF's F to compute probabilities: $F(b) - F(a) = \int_a^b f(x)\,dx$.

One can compute **area**, **volume**, **cumulative distribution functions** using integrals.

To determine the **catastrophes** for a family $f_c(x)$ of functions, determine the critical points in dependence of c and find values c, where a critical point changes from a local minimum to a local maximum.

Important integrals

Which one is the derivative which the integral?

$\sin(x)$	$-\cos(x)$.	$\log(x)$	$x\log(x) - x$
$\tan(x)$	$1/\cos^2(x)$.	$1/x$	$\log(x)$
$\arctan(x)$	$1/(1+x^2)$.	$-1/(1+x^2)$	$\mathrm{arccot}(x)$
$1/\sqrt{1-x^2}$	$\arcsin(x)$	$-1/\sqrt{1-x^2}$	$\arccos(x)$

Improper integrals

$\int_1^\infty 1/x^2\,dx$ Prototype of improper integral which exists.
$\int_1^\infty 1/x\,dx$ Prototype of improper integral which does not exist.
$\int_0^1 1/x\,dx$ Prototype of improper integral which does not exist.
$\int_0^1 1/\sqrt{x}\,dx$ Prototype of improper integral which does exist.

The fundamental theorem

$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$

$\int_0^x f'(t)\,dt = f(x) - f(0)$.

This implies

$\int_a^b f'(x)\, dx = f(b) - f(a)$

Without limits of integration, we call $\int f(x)\, dx$ the **anti derivative**. It is defined up to a constant. For example $\int \sin(x)\, dx = -\cos(x) + C$.

PDF and CDF

Calculus applies directly if there are situations where one quantity is the derivative of the other.

function	anti derivative
probability density function	cumulative distribution function

Most important integrals

The most important integral is the integral

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

holds for all n different from 1.

$$\int \frac{1}{x}\, dx = \log(x)$$

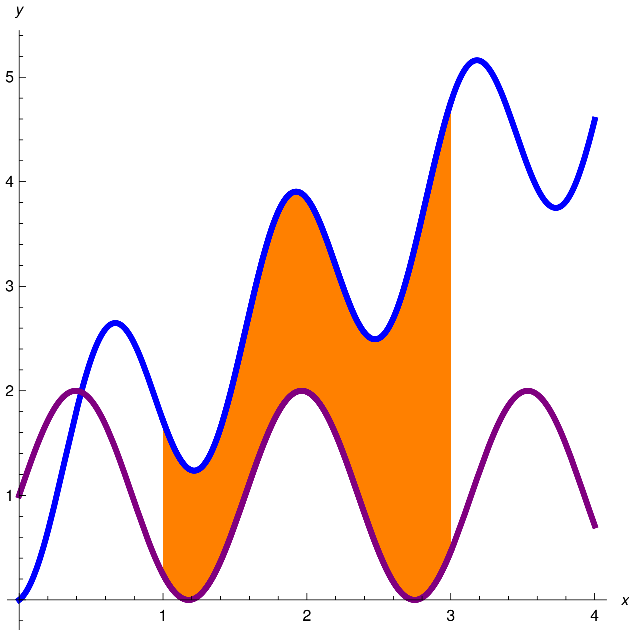
Example: $\int \sqrt{x+7}\, dx = \frac{2}{3}(x+7)^{3/2}$.

Example: $\int \frac{1}{x+5}\, dx = \log(x+5)$

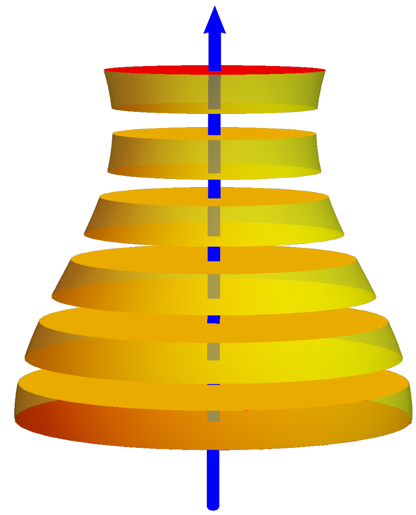
Example: $\int \frac{1}{4x+3}\, dx = \log(4x+3)/4$

Key pictures

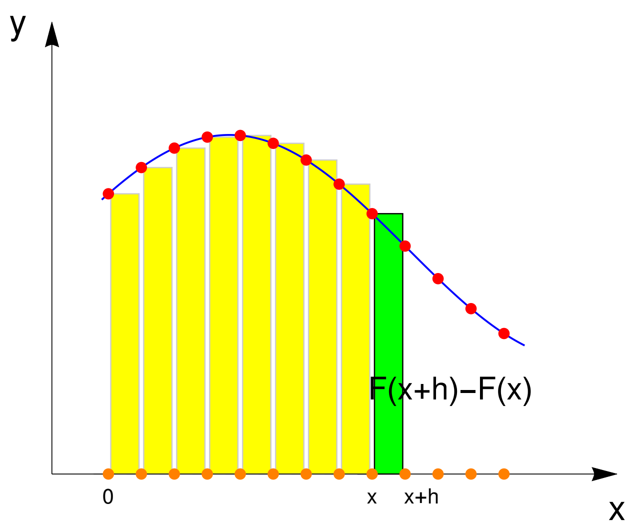
$$\boxed{\text{Area}} = \int_a^b (g(x) - f(x))\, dx$$



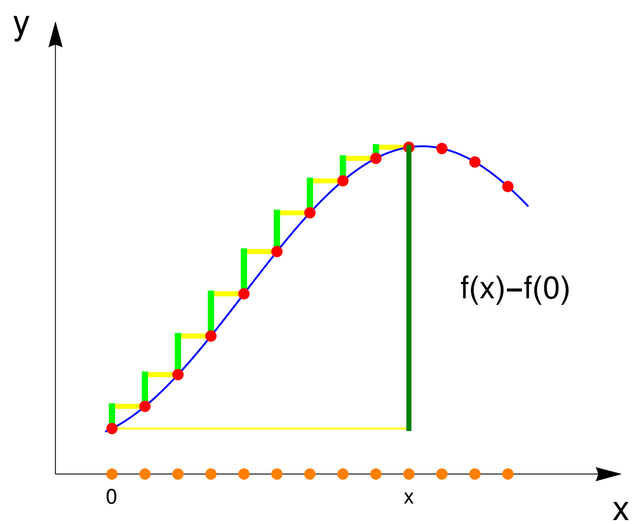
$$\boxed{\text{Volume}} = \int_a^b A(z)\, dz$$



$$\boxed{f(x)} = \frac{d}{dx} \int_0^x f(t)\, dt$$



$$\boxed{f(x) - f(0)} = \int_0^x f'(t)\, dt$$



Make a picture, whenever we deal with an area or volume computation! For volume computations, just integrate area of the cross section $A(x)$. For area computations integrate $g(x) - f(x)$, where $g(x)$ is above $f(x)$.

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