INTRODUCTION TO CALCULUS

MATH 1A

Unit 22: Improper Integrals

Lecture

22.1. In this lecture, we look at integrals on infinite intervals or integrals, where the function can get infinite at some point. These integrals are called **improper integrals**. The area under the curve can remain finite or become infinite.

Example: What is the integral

$$\int_1^\infty \frac{1}{x^2}\, dx\ ?$$

Since the anti-derivative is $-1/x$, we have

$$\frac{-1}{x}|_1^\infty = -1/\infty + 1 = 1\ .$$

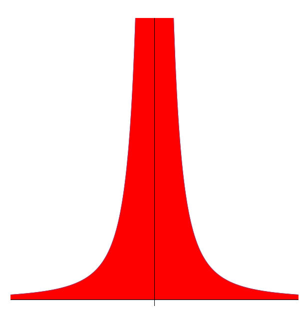
To justify this, compute the integral $\int_1^b 1/x^2 dx = 1 - 1/b$ and see that in the limit $b \to \infty$, the value 1 is achieved.

22.2. In a previous lecture, we have seen a chocking example similar to the following one:

Example:

$$\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_{-1}^1 = -1 - 1 = -2\ .$$

This does not make any sense because the function is positive so that the integral should be a positive area. The problem is this time not at the boundary $-1, 1$. The sore point is $x = 0$ over which we have carelessly integrated over.



22.3. The next example illustrates the problem with the previous example better:
Example: The computation

$$\int_0^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_0^1 = -1 + \infty \; .$$

indicates that the integral does not exist. We can justify by looking at integrals

$$\int_a^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_a^1 = -1 + \frac{1}{a}$$

which are fine for every $a > 0$. But this does not converge for $a \to 0$. Now

$$\int_{-1}^{-a} \frac{1}{x^2}\, dx = -\frac{1}{x}|_{-1}^{-a} = -1 + \frac{1}{a} \; .$$

If we add up both, we get $-2 + 2/a$. This value is positive for every $0 < a < 1$ but it does not disappear for $a \to 0$.

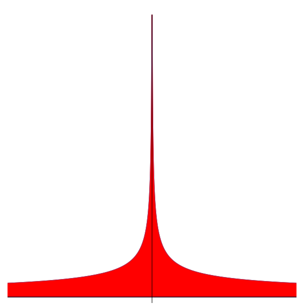
22.4. Do we always have a problem if the function goes to infinity at some point?
Example: Find the following integral

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx \; .$$

Solution: Since the point $x = 0$ is problematic, we integrate from a to 1 with positive a and then take the limit $a \to 0$. Since $x^{-1/2}$ has the anti-derivative $x^{1/2}/(1/2) = 2\sqrt{x}$, we have

$$\int_a^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2(1 - \sqrt{a}) \; .$$

There is no problem with taking the limit $a \to 0$. The answer is 2. Even so the region is infinite its area is finite. This is an interesting example. Imaging this to be a container for paint. We can fill the container with a finite amount of paint but the wall of the region has infinite length.



Example: Evaluate the integral $\int_0^1 1/\sqrt{1-x^2}\, dx$. **Solution:** The anti-derivative is $\arcsin(x)$. In this case, it is not the point $x = 0$ which produces the difficulty. It is the point $x = 1$. Take $a > 0$ and evaluate

$$\int_0^{1-a} \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x)|_0^{1-a} = \arcsin(1-a) - \arcsin(0) \; .$$

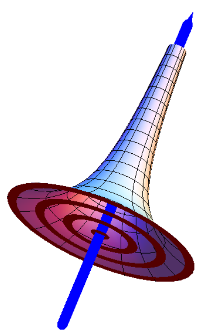
Now $\arcsin(1-a)$ has no problem at limit $a \to 0$. Since $\arcsin(1) = \pi/2$ exists. We get therefore the answer $\arcsin(1) = \pi/2$.

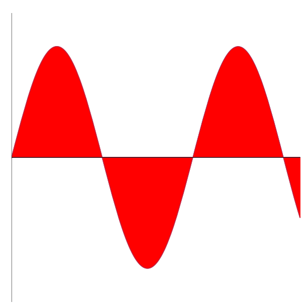
Example: Rotate the graph of $f(x) = 1/x$ around the x-axes and compute the volume of the solid between 1 and ∞. The cross section area is π/x^2. If we look at the integral from 1 to a fixed R, we get

$$\int_1^R \frac{\pi}{x^2}\, dx = -\frac{\pi}{x}|_1^R = -\pi/R + \pi \; .$$

This converges for $R \to \infty$. The volume is π. This famous solid is called **Gabriel's trumpet**. This solid is so prominent because if you look at the surface area of the small slice, then it is larger than $dx 2\pi/x$. The total surface area of the trumpet from 1 to R is therefore larger than $\int_1^R 2\pi/x\, dx = 2\pi(\log(R) - \log(1))$. which goes to infinity. We can **fill** the trumpet with a finite amount of paint but we can not **paint** its surface.

Example: Evaluate the integral $\int_0^\infty \sin(x)\, dx$. **Solution.** There is no problem at the boundary 0 nor at any other point. We have to investigate however, what happens at ∞. Therefore, we look at the integral $\int_0^b \sin(x)\, dx = -\cos(x)|_0^b = 1 - \cos(b)$. We see that the limit $b \to \infty$ does not exist. The integral fluctuates between 0 and 2.





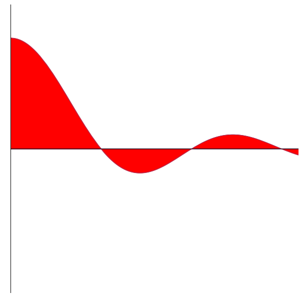
22.5. The next example leads to a topic in a follow-up course. It is not covered here, but could make you curious:

Example: What about the integral

$$I = \int_0^\infty \frac{\sin(x)}{x}\, dx \; ?$$

Solution. The anti derivative is the Sine integral $Si(x)$ so that we can write $\int_0^b \sin(x)/x\, dx = Si(b)$. It turns out that the limit $b \to \infty$ exists and is equal to $\pi/2$ but this is a topic for a second semester course like Math 1b. The integral can be written as an alternating series, which converges and there are many ways to compute it: [1]

[1]Hardy, Mathematical Gazette, 5, 98-103, 1909.



22.6.

$\int_a^\infty f(x)\,dx$ id defined as $\lim_{t\to\infty}\int_a^t f(x)\,dx$ if the limit exists.
$\int_0^b f(x)\,dx$ is defined as $\lim_{t\to 0}\int_t^b f(x)\,dx$ if the limit exists

Homework

Problem 22.1: Evaluate the integral $\int_0^1 x^{2/3}\,dx$.

Problem 22.2: For which $0 < p < \infty$ does the integral $\int_1^\infty 1/x^p\,dx$ exist? To investigate this, look at $\int_1^t 1/x^p\,dx$ and decide in which case the limit $t \to \infty$ exists.

Problem 22.3: Evaluate the improper integral $\int_{-1}^1 1/\sqrt{1-x^2}\,dx$. This example is related to the arcsin distribution in probability theory. Guess where this name come from?

Problem 22.4: Evaluate the integral $\int_{-3}^4 (x^2)^{1/3}\,dx$. To make sure that the integral is fine, check separately whether $\int_{-3}^0$ and $\int_0^4$ work.

The integral $\int_{-2}^1 1/x\,dx$ does not exist. We can however take a positive $a > 0$ and look at

$$\int_{-2}^{-a} 1/x\,dx + \int_a^1 1/x\,dx = \log|a| - \log|-2|) + (\log|1| - \log|a|) = \log(2)\ .$$

If the limit exists, it is called the **Cauchy principal value** of the improper integral.

Problem 22.5: Find the Cauchy principal value of

$$\int_{-4}^5 3/x^3\,dx\ .$$