INTRODUCTION TO CALCULUS

MATH 1A

Unit 17: Riemann Integral

Lecture

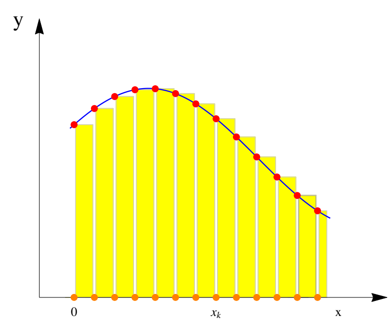
17.1. In this lecture, we define the definite integral $\int_0^x f(t)\ dt$ if f is a differentiable function. We then compute it for some basic functions. We have previously defined the **Riemann sums**

$$Sf(x) = h[\ f(0) + f(h) + f(2h) + \cdots + f(kh)]\ ,$$

where k is the largest integer such that $kh < x$. Lets write S_n if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the **Riemann integral** as the limit of these sums $S_n f$, when the **mesh size** $h = 1/n$ goes to zero.

Definition: Define

$$\int_0^x f(t)\ dt = \lim_{n\to 0} S_n f(x)\ .$$



For any differentiable function, the limit exists.

Proof: Assume first $f \geq 0$ on $[0, x]$. Let M be such that $f \leq M$ and $f' \leq M$ on $[0, x]$. The Riemann sum $S_n f(x)$ is the total area of K rectangles. Let $Sf(x)$ denote the area under the curve. If M is the maximal slope of f on $[0, x]$, then on each interval $[j/n, (j + 1)/n]$, we have $|f(x) - f(j/n)| \leq M/n$ so that the area error is smaller than M/n^2. As there are n such errors the error is smaller than $M(M/n^2) = M/n$. An additional rectangle above $[Kh, x]$ of area $\leq M/n$ is an upper bound on the discrepancy at the right boundary. If we add all the $k \leq xn$ "roof area errors" and the "side area" up, we get so

$$Sf(x) - S_n f(x) \leq \frac{kM}{n^2} + \frac{M}{n} \leq \frac{xnM}{n^2} + \frac{M}{n} = \frac{xM + M}{n}\ .$$

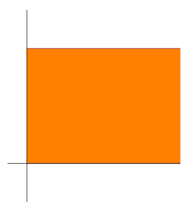
This converges to 0 for $n \to \infty$. The limit is therefore the area $Sf(x)$. For a general, not necessarily non-negative function, we write $f = g - h$, where g, h are non-negative and have $\int_0^x f(x)\,dx = \int_0^x g(x)\,dx - \int_0^x h(x)\,dx$.

For non-negative f, the value $\int_0^x f(x)\,dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

17.2. Remark: the Riemann integral is defined here as the limit $h\sum_{x_k=kh\in[0,x)} f(x_k)$. It converges to the area under the curve for all **continuous** functions but since we work with differentiable functions in calculus we restricted to that. Not **all** bounded functions can be integrated naturally like this. There are discontinuous functions like the **salt and pepper function** which is defined to be $f(x) = 1$ if x is rational and zero else. Now $Sf(x) = 1$ for rational h and $Sf(x) = 0$ if h is irrational. Therefore, an other integral, the **Lebesgue integral** is used too: it can be defined as the limit $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ where x_k are **random points** in $[0, x]$. This **Monte-Carlo integral** definition of the Lebesgue integral gives the integral 0 for the salt and pepper function because rational numbers have zero probability.

17.3. Remark: The Riemann integral can be defined for partitions $x_0 < x_1 < \cdots < x_n$ of points of the interval $[0, x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}). For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated.

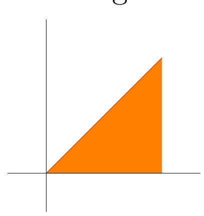
Example: If $f(x) = c$ is constant, then $\int_0^x f(t)\,dt = cx$. We can see also that $cnx/n \leq S_n f(x) \leq c(n+1)x/n$.



Example: Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Adding up the Riemann sum is more difficult. Let k be the largest integer smaller than $xn = x/h$. Then

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2}\ .$$

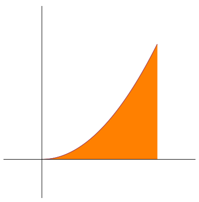
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\,dt = cx^2/2$.

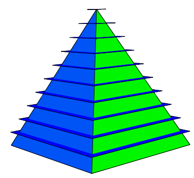


Example: Let $f(x) = x^2$. In this case,we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\, dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2\, dt = \frac{x^3}{3} .$$

This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section. Think about $t^2 dt$ as a slice of the pyramid of area t^2 and height dt. Adding up the volumes of all these slices gives the volume.

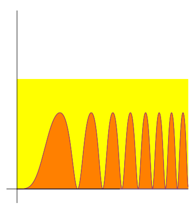




Linearity of the integral (see homework) $\int_0^x f(t) + g(t)\, dt = \int_0^t f(t)\, dt + \int_0^x g(t)\, dt$ and $\int_0^x \lambda f(t)\, dt = \lambda \int_0^x f(t)\, dt$.

Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_0^x f(t)\, dt \leq Mx$.

Example: $\int_0^x \sin^2(\sin(\sin(t)))/x\, dt \leq x$. **Solution**. The function $f(t)$ inside the interval is nonnegative and smaller or equal to 1 The graph of f is therefore contained in a rectangle of width x and height 1.



We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x) - g(x)| \leq c$, then

$$\int_0^x |f(x) - g(x)|\, dx \leq cx .$$

We know identities like $S_n [x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$.
The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$. This gives us:

$\int_0^x t^n\, dt = \frac{x^{n+1}}{n+1}$

$\int_0^x e^t\, dt = e^x - 1$

$\int_0^x \cos(t)\, dt = \sin(x)$

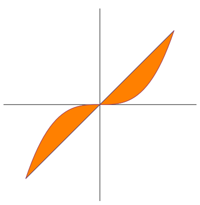
$\int_0^x \sin(t)\, dt = 1 - \cos(x)$

Homework

In the following homework you can use that $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ if F is a function which satisfies $F'(x) = f(x)$. We have already verified it for sums.

Problem 17.1: a) What is the integral $\int_1^2 x^8\,dx$?
b) Find the integral $\int_0^1 8t^7 + e^t\,dt$.
c) Calculate $\int_{-1}^1 \frac{1}{1+x^2}\,dx$.
d) Find $\int_0^{\pi/2} \sin^2(t)\,dt$. e) Find $\int_0^{\pi/2} \sin^4(t)\,dt$.

Problem 17.2: The region enclosed by the graph of x and the graph of x^3 has a propeller type shape as seen in the picture. Find its (positive) area.



Problem 17.3: Make a geometric picture for each of the following statements (which are rules for integration):

- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
- $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$.
- $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$.

Problem 17.4: Here are some more challenging integrals. Maybe you have to guess:
a) $\int_0^1 (3/2)\sqrt{1+x}\,dx$
b) $\int_0^{\sqrt{\log(2)}} 16xe^{-x^2}\,dx$
c) $\int_0^{\pi} \sin^4(x)\,dx$
d) $\int_1^e 5\log(x)/x\,dx$ For c), use double angle formulas, twice.

Problem 17.5: In this problem, it is crucial that you plot the function first. Split the integral up into parts.
a) Find $\int_0^3 |x-1|\,dx$. Distinguish cases.
b) Find $\int_0^3 f(x)\,dx$ for $f(x) = |x - |x-1||$. Also here, distinguish cases