INTRODUCTION TO CALCULUS

MATH 1A

Unit 16: Catastrophes

Lecture

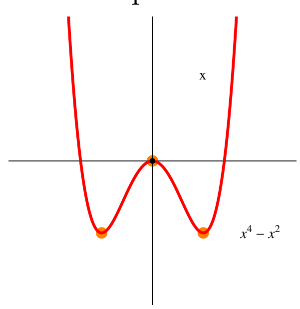
16.1. In this lecture, we are interested how minima and maxima change when a parameter changes. Nature, economies, processes favor extreme points. If optimal values change smoothly with parameters, how come that the outcome is often not smooth? What is the reason that an economic change can go so fast, once a tipping point is reached? One can explain this with mathematical models. A simple example explains the general principle:

If a local minimum disappears to some change of external parameter, the system settles in a new stable equilibrium. The new equilibrium can be far away from the original one.

16.2. To see this, let us look at the following optimization problem

Example: Find all the extrema of the function

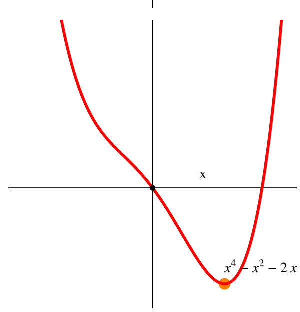
$$f(x) = x^4 - x^2$$



Solution: $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.

Example: Now find all the extrema of the function

$$f(x) = x^4 - x^2 - 2x$$



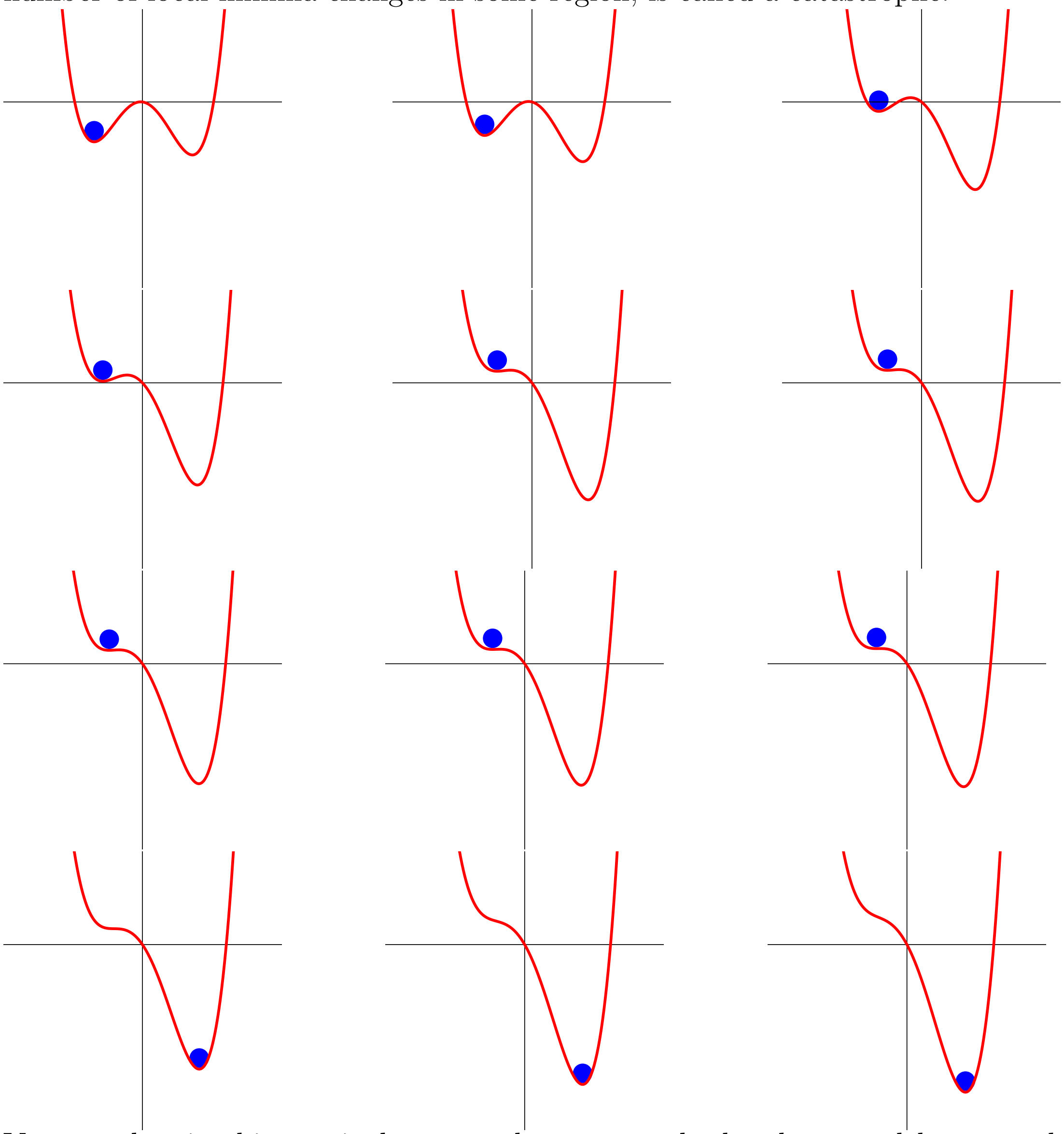
There is only one critical point. It is $x = 1$.

16.3. When the first graph is morphed to the second example, the local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is a **parameter**. We look at the position of the first critical point of the function. Catastrophe theorists look at the following **assumption**:

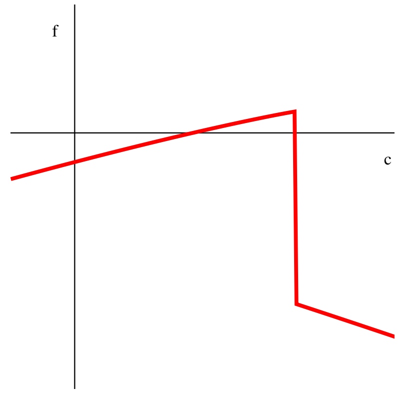
Definition: A **stable equilibrium** is a local minimum of the function. Assume the system depends on a parameter, then the minimum depends on this parameter. It remains a stable equilibrium until it disappears. If that happens, the system settles in a neighboring stable equilibrium.

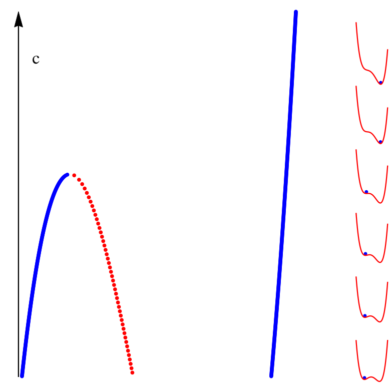
Definition: A parameter value c_0 at which somewhere a stable minimum disappears, is called a **catastrophe**. In other words, if for $c < c_0$ a different collection of local minima exist than for $c > c_0$, then the parameter value c_0 is called a **catastrophe**.

16.4. In order to visualize a catastrophe, we draw the graphs of the function $f_c(x)$ for various parameters c and look at the local minima. At a parameter value, where the number of local minima changes in some region, is called a catastrophe.



You see that in this particular case, the catastrophe has happened between the 9th and 10th picture. Here is the position of the equilibrium point in dependence of c.





16.5. A **bifurcation diagram** displays the equilibrium points as they change in dependence of the parameter c. The vertical axes is the parameter c, the horizontal axes is x. At the bottom for $c = 0$, there are three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum. Here is an important principle:

Catastrophes often lead to a strict and abrupt decrease of the minimal critical value. It is not possible to reverse the process in general.

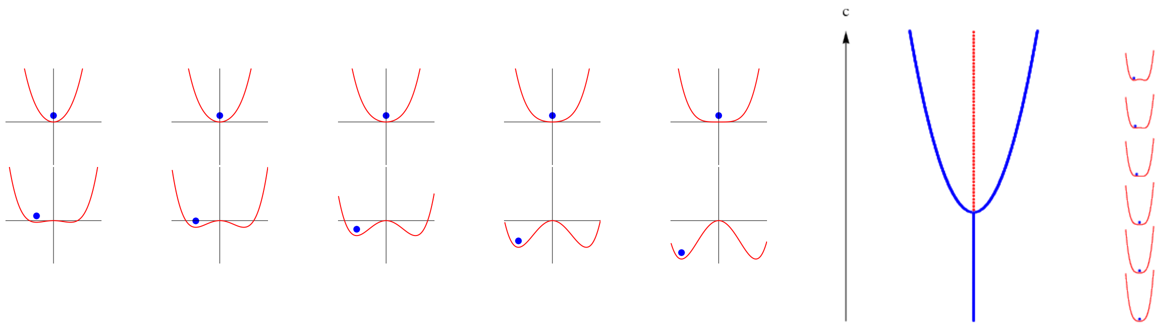
If we look at the above "movie" of graphs and run it backwards and use the same principle, we do not end up at the position we started with. The new equilibrium remains the equilibrium nearby.

Catastrophes are in general **irreversible**.

16.6. We know this from experience: it is easy to screw up a relationship, get sick, have a ligament torn or lose trust. Building up a relationship, getting healthy or gaining trust usually happens continuously and slowly. Ruining the economy of a country or a company or losing a good reputation of a brand can be quick. It takes time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

Example: We look at the example $f(x) = x^4 - cx^2$ with $-1 \leq c \leq 1$ in class.



Homework

In this homework, we study a catastrophe for the function

$$f(x) = x^6 - x^4 + cx^2 ,$$

where c is a parameter between 0 and 1.

Problem 16.1: a) Find all the critical points in the case $c = 0$ and analyze their stability. b) Find all the critical points in the case $c = 1$ and analyze their stability.

Problem 16.2: Plot the graph of the function $f(x)$ for 10 values of c between 0 and 1. You can use a graphing calculator or Wolfram alpha. Mathematica example code is below.

Problem 16.3: If you change from $c = -0.3$ to 0.6, pinpoint the value for the catastrophe and show a rough plot of $c \to f(x_c)$, the value at the first local minimum x_c in dependence of c. The text above provides this graph for an other function. It is the graph with a discontinuity.

Problem 16.4: If you change back from $c = 0.6$ to -0.3 pinpoint the value for the catastrophe. It will be different from the one in the previous question.

Problem 16.5: Sketch the bifurcation diagram. That is, if $x_k(c)$ is the k'th equilibrium point, then draw the union of all graphs of $x_k(c)$ as a function of c (the c-axes pointing upwards). As in the two example provided, draw the local maximum with dotted lines.

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Manipulate[ Plot[x^6 - x^4 + c x^2, {x, -1, 1}], {c, 0, 1}]
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