INTRODUCTION TO CALCULUS

MATH 1A

Unit 10: Chain rule

Lecture

10.1. If we want to take the derivative of a composition of functions like $f(x) = \sin(x^7)$, the product rule does not work. The functions are not multiplied but are "chained" in the sense that we evaluate first x^7 then apply sin to it. In order to differentiate, we take the derivative of the x^7 then multiply this with the derivative of the function sin evaluated at x^7. The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \; .$$

10.2.

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} \; .$$

Write $H(x) =$ g(x+h)-g(x) in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} \; .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit.

10.3. Let us look at some examples.

Example: Find the derivative of $f(x) = (4x^2 - 1)^{17}$. **Solution** The inner function is $g(x) = 4x^2 - 1$ with derivative $8x$. We get therefore $f'(x) = 17(4x - 1)^6 \cdot 8x$. Remark. We could have expanded out the power $(4x^2 - 1)^{17}$ first and avoided the chain rule. Try it. You will see that the rule of avoiding the **chain rule** is called the **pain rule**.

Example: Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

Example: For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx + d) + b = acx + ad + b$ which has the derivative ac. This agrees with the definition of f times the derivative of g. You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

10.4. One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

Example: Find the derivative of the natural logarithm function $\log(x)$. [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x))\log'(x) = x\log'(x) = 1$. Therefore $\log'(x) = 1/x$.

$\frac{d}{dx}\log(x) = 1/x$.

Definition: Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$.

Example: Find the derivative of $\arcsin(x)$. **Solution**. We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} .$$

Example: Find the derivative of $\arccos(x)$. **Solution**. We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} .$$

Example: $f(x) = \sin(x^2+3)$. Then $f'(x) = \cos(x^2+3)2x$.

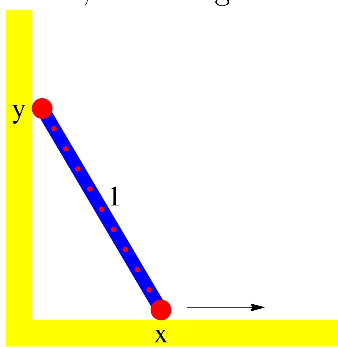
Example: $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$.

Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

Example: Let us compute the derivative of $\sin(\sqrt{x^5-1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5-1}$ gives $\cos(\sqrt{x^5-1})\frac{d}{dx}\sqrt{x^5-1}$. Apply now the chain rule again for the derivative on the right hand side.

[1] We always write $\log(x)$ for the natural log. Similarly as $\exp(x) = e^x$, one can also use ln which stands for "logarithmus naturalis". Practically all computer languages like Python, C, Perl, R, Matlab, Mathematica use log. Paul Halmos called "ln" a childish notation which no mathematician ever used.

Example: Here is a famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1-x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) = -2x/\sqrt{1-x^2}$.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

Example: For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))))$.

Example: Take the derivative of $f_3(x) = e^{e^{e^x}}$.

Solution We can also write this as $\exp(\exp(\exp(x))$. The derivative is

$$\exp(\exp(\exp(x)))\,\exp(\exp(x))\,\exp(x)\ .$$

Example: Lets push that to the extreme and differentiate

$$f(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x)))))))))))$$

Here is the poetic formula obtained when running this in Mathematica:

$$\mathrm{F[f\_] := Exp[f]; D[Last[NestList[F, x, 11]], x]}$$

$$\exp\left(e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{x}}}}}}} + e^{e^{e^{e^{e^{e^{x}}}}}} + e^{e^{e^{e^{e^{x}}}}} + e^{e^{e^{e^{x}}}} + e^{e^{e^{x}}} + e^{e^{x}} + e^{x} + x\right)$$

Example: Find the derivative of $1/\sin(x)$ using the quotient rule.

Solution $-\cos(x) \cdot 1/\sin^2(x)$.

Example: Find the derivative of $f(x) = 1/\sin(x)$ using the chain rule.

Solution. The outer function is $f(x) = 1/x$. Therefore $f'(x) = -\cos(x)/\sin^2(x)$.

Homework

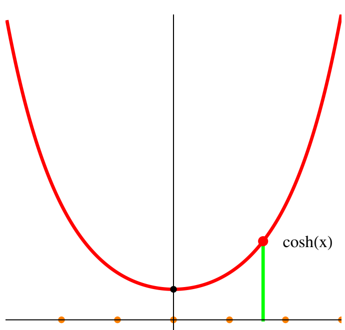
Problem 10.1: Find the derivatives of the following functions:
a) $f(x) = \sin(\log(x))$
b) $f(x) = \tan(x^{11})$
c) $f(x) = \exp(1/(1+x^2))$
d) $(3+\sin(x))^{-5}$

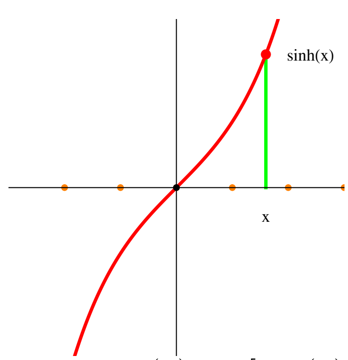
Problem 10.2: Find the derivatives of the following functions at $x = 1$.
a) $f(x) = -x\log(x)$. (where log is natural log)
b) $\sqrt{x^5+1}$
c) $(1+x^2+x^4)^{100}$
d) $\frac{5x^4}{2\sqrt{x^5+1}}$

Problem 10.3: a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$ and using the product rule.
b) Find the derivative of $f(x) = \text{arccot}(x)$ by differentiating $\cot(\text{arccot}(x)) = x$ and using the chain rule.

Problem 10.4: a) Find the derivative of $\sqrt{x}$ by differentiating the identity $f(x)^2 = x$ leaving f as it is and solving for $f'(x)$.
b) Find the derivative of $x^{m/n}$ by differentiating the identity $f(x)^n = x^m$ leaving f as it is and solving for $f'(x)$.

Problem 10.5: a) Find the derivative of the inverse $\text{arccosh}(x)$ of $\cosh(x)$ by using the chain rule.
b) Find the derivative of the inverse $\text{arcsinh}(x)$ of $\sinh(x)$ by using the chain rule.





Define $\cosh(x) = [\exp(x)+\exp(-x)]/2$ and $\sinh(x) = [\exp(x)-\exp(-x)]/2$. the **hyperbolic cosine** and **hyperbolic sine**. The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine. You check $\cosh^2(x) - \sinh^2(x) = 1$.