INTRODUCTION TO CALCULUS

MATH 1A

Unit 6: Fundamental theorem

LECTURE

6.1. Calculus is a theory of **differentiation** and **integration**. We explore here this concept again in a simple setup and practice differentiation and integration **without taking limits**. We fix a positive constant h and take differences and sums. The fundamental theorem of calculus for $h = 1$ generalizes. We can then differentiate and integrate polynomials, exponentials and trigonometric functions. Later, we will do the same with actual derivatives and integrals. But now, we can work with arbitrary continuous functions. The constant h never pops up. Thinking of it as something fixed, like the God-given **Planck constant** $1.6 \cdot 10^{-35}m$. In the standard calculus of Newton and Leibniz the limit $h \to 0$ is taken.

Definition: Given $f(x)$, define the **difference quotient**

$$\boxed{Df(x) = \frac{f(x+h)-f(x)}{h}}$$

6.2. If f is continuous then Df is a continuous. For shorthand, we call it simply the "derivative". We keep h constant and positive here. As an example, lets take the **constant function** $f(x) = 5$. We get $Df(x) = (f(x+h) - f(x))/h = (5-5)/h = 0$ everywhere. You can see that in general, if f is a constant function, then $Df(x) = 0$.

6.3. $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if $f(x) = mx + b$, then $Df(x) = \boxed{m}$.

For constant functions, $Df(x) = 0$. For linear functions, $Df(x)$ is the slope.

6.4. For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h = \boxed{2x + h}$.

6.5. Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \leq jh < x$:

Definition: Given $f(x)$ define the **Riemann sum**

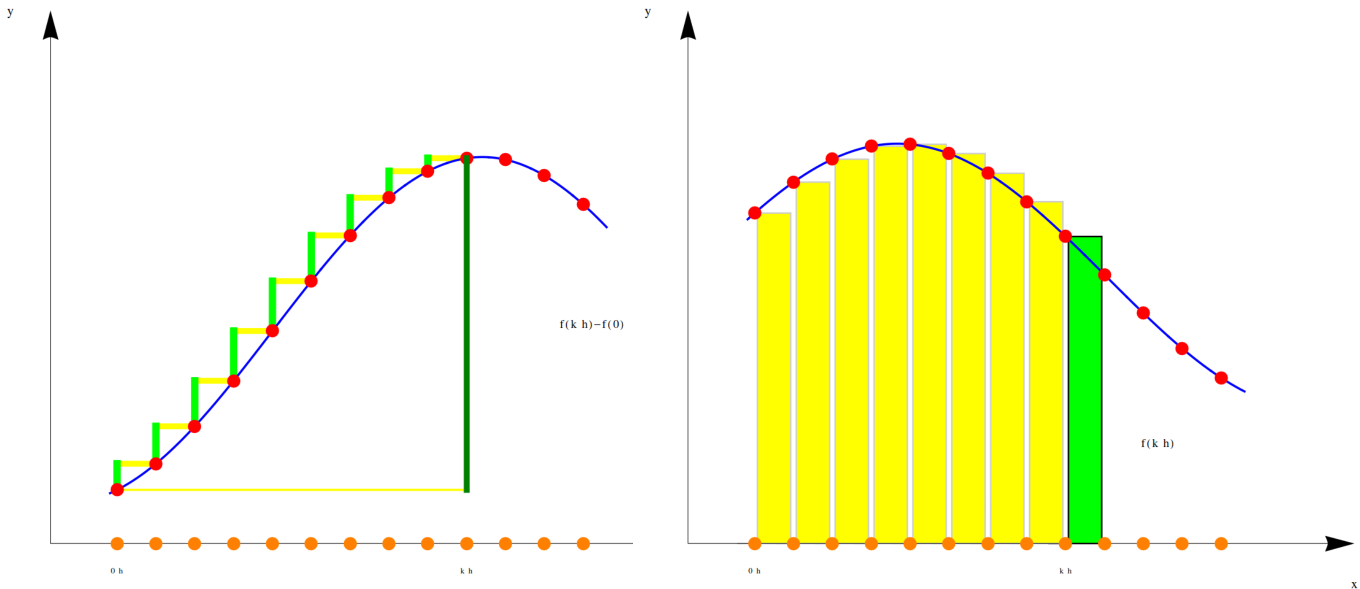
$$\boxed{Sf(x) = h[\ f(0) + f(h) + f(2h) + \cdots + f((k-1)h)\ ]}$$

In short hand, we call Sf also the "integral" or "anti-derivative" of f. It will become the integral in the limit $h \to 0$ later in the course.

6.6. Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \leq h$, and $Sf(x) = h$ for $h \leq x < 2h$ and $Sf(x) = 2h$ for $2h \leq x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference $Df(x)$ will become the **derivative** $f'(x)$.
The sum $Sf(x)$ will become the **integral** $\int_0^x f(t)\,dt$.

Df means rise over run and is close to the slope of the graph of f.
Sf means areas of rectangles and is close to the area under the graph of f.



6.7. Here is the **quantum fundamental theorem of calculus**

Theorem: Sum the differences and get

$$SDf(kh) = f(kh) - f(0)$$

Theorem: Difference the sum and get

$$DSf(kh) = f(kh)$$

Example: For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$f(x+h)-f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h)-(x-mh+h)) = [x]^{m-1}hm$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. We have obtained the important formula $\boxed{D[x]^m = m[x]^{m-1}}$

6.8. We can establish from this differentiation formulas for **polynomials**. We will leave away the square brackets later to make it look like the calculus we will do later on. In the homework, we already use the usual notation.

6.9. If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$. The fundamental theorem allows us to integrate and get $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Definition: Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero.

As a rescaled exponential, it is continuous and monotone. Indeed, using rules of the logarithm we can see $\exp_h(x) = e^{x(\log(1+h)/h)} = e^{xA}$. It is actually a classical exponential with some constant A.

6.10. The function $\exp_h(x) = (1+h)^{x/h}$ has the property that its derivative is the function again (see unit 4). We also have $\exp_h(x+y) = \exp_h(x)\exp_h(y)$. More generally, define $\exp(a \cdot x) = (1+ah)^{x/h}$. It satisfies $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$ We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$. For now, only the differentiation rule for this function is important.

6.11. If a is replaced with ai where $i = \sqrt{-1}$, we have $\exp(1+ia)(1+aih)^{x/h}$ and still $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Taking real and imaginary parts define new **trig functions** $\exp_h^{ai}(x) = \cos_h(a \cdot x) + i\sin_h(a \cdot x)$. These functions are real and morph into the familiar cos and sin functions for $h \to 0$. For any $h > 0$ and any a, we have now $\boxed{D\cos_h(a \cdot x) = -a\sin_h(a \cdot x) \text{ and } D\sin_h(a \cdot x) = a\cos_h(a \cdot x).}$ We will later derive these identities for the usual trig functions.

6.12.

Definition: Define $\log_h(x)$ as the inverse of $\exp_h(x)$ and

$$1/[x+a]_h = D\log_h(x+a).$$

6.13. We have directly from the definition $\boxed{S1/[x+1]_h = \log_h(x+1)}$ As a consequence we can compute things like

$$S\frac{1}{[3x+3]} = \frac{1}{3}S\frac{1}{[x+1]} = \frac{1}{3}\log_h(x+1)\ .$$

More generally $S(1/[x+a]) = \log(x+a) - \log(a)$.

Homework

Use the differentiation and integration rules to find.

Problem 6.1: Find the derivatives $Df(x)$ of the following functions:

a) $f(x) = x^{11} - 3x^4 + 5x + 1$
b) $f(x) = -x^4 + 8\log(x)$
c) $f(x) = -3x^3 + 17x^2 - 5x$.
d) $f(x) = \log(x+5)$.

Problem 6.2: Find the integrals $Sf(x)$ of the following functions assuming $Sf(0) = 0$:

a) $f(x) = x^9 - 5$.
b) $f(x) = x^2 + 6x^7 - x$
c) $f(x) = -3x^3 + 17x^2 - 5x$
d) $f(x) = \exp(5x) + \sin(9x)$

Problem 6.3: Find the derivatives $Df(x)$ of the following functions
a) $f(x) = \exp(9 \cdot x + 3) + x^6$ b) $f(x) = 8\exp(-3 \cdot x) + 9x^6$
c) $f(x) = \exp(6 \cdot x) + log/(1 + x)$ d) $\log(1 - x^2)$

Problem 6.4: a) Assume $h = 1/100$. Use Wolfram alpha to plot $\cos_h(x)$ and $\sin_h(x)$ on the interval $[-2\pi, 2\pi]$. **Hint.** This means you have to plot the real and imaginary part of $(1 + i/100)^{100x}$. If you enter the expression into Wolfram alpha, it will plot the real and imaginary part.
b) Do the same for $h = 1/1000$. What has changed?

Problem 6.5: a) Write down again on your own that if $f(x) = (1+h)^{x/h}$, then $Df(x) = f(x)$. (We have done this already twice. Do it again!).
b) Write down again on your own that if $f(x) = x(x-h)(x-2h)(x-3h)$, then $Df(x) = 4x(x-h)(x-2h)(x-3h)$ meaning $D[x]^4 = 4[x]^3$.

6.14. In this unit we used a self-contained calculus which does not refer to any limit. It is **quantum calculus** and like the calculus we will see when $h \to 0$, where the formulas are **all the same**. So, why do we not stop here, call it a day, and have an extended spring break until May 2020? Because we want also to have fun in the limit $h \to 0$!

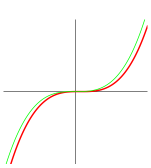
Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

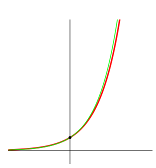
Differentiation rules
$Dx^n = nx^{n-1}$
$De^{a \cdot x} = ae^{a \cdot x}$
$D\cos(a \cdot x) = -a\sin(a \cdot x)$
$D\sin(a \cdot x) = a\cos(a \cdot x)$
$D\log(x + a) = 1/(x + a)$

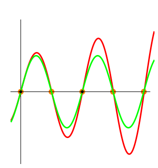
Integration rules (for $x = kh$)
$Sx^n = x^{n+1}/(n + 1)$
$Se^{a \cdot x} = (e^{a \cdot x} - 1)/a$
$S\cos(a \cdot x) = \sin(a \cdot x)/a$
$S\sin(a \cdot x) = -\cos(a \cdot x)/a$
$S\frac{1}{x+a} = \log(x + a) - \log(a)$

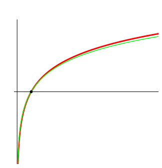
Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x + h)$.

Pictures


$[x]_h^3$ for $h = 0.1$


$\exp_h(x)$ for $h = 0.1$


$\sin_h(x)$ for $h = 0.1$


$\log_h(x)$ for $h = 0.1$

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