INTRODUCTION TO CALCULUS

MATH 1A

Unit 5: Intermediate value theorem

Lecture

5.1. The problem to find solutions to equations can be reduced to finding roots.

Definition: If $f(a) = 0$, then a is called a **root** of f. For $f(x) = \cos(2x)$ for example, there are roots at $x = \pi$ or $x = 3\pi$ or $x = -\pi$.

5.2. Here are a few examples

Example: Find the roots of $f(x) = 4x + 6$. **Answer**: we set $f(x) = 0$ and solve for x. In this case $4x + 6 = 0$ and so $x = -3/2$.

Example: Find the roots of $f(x) = x^2 + 2x + 1$. **Answer**: Because $f(x) = (x + 1)^2$ the function has a root at $x = -1$.

Example: Find the roots of $f(x) = (x - 2)(x + 6)(x + 3)$. **Answer:** Since the polynomial is factored already, it is easy to see the roots $x = 2, x = -6, x = -3$.

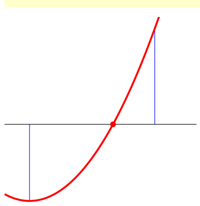
Example: $f(x) = 12 + x - 13x^2 - x^3 + x^4$. Find the roots of f. There is no formula. Just try ($12 = 3 * 4$ is a hint). We see $x = 1, x = -3, x = 4, x = -1$ are the roots.

Example: The function $f(x) = \exp(x)$ does not have any root.

Example: The function $f(x) = \log|x| = \ln|x|$ has roots $x = 1$ and $x = -1$.

Example: $f(x) = 2^x - 16$ has the root $x = 2$.

Intermediate value theorem of Bolzano. If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b).

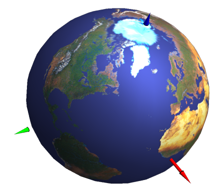


5.3. The proof is constructive: we can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at the point $c = (a + b)/2$. If $f(c) < 0$, then look take $[c, b]$ as your new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2^{-n}(b - a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and that $f(u_n), f(v_n)$ have different signs. Both point sequences u_n, v_n converge to a root of f.

Example: Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root.
Solution. The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. Use the theorem.

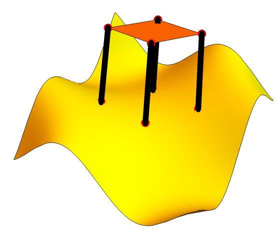
Example: There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem.

Example: **Earth Theorem.** There is a point on the earth, where temperature and pressure agrees with the temperature and pressure on the antipode.



Proof. Draw a meridian through the poles and let $f(x)$ be the temperature on that circle. Define $g(x) = f(x) - f(x + \pi)$. If this function is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. There exists therefore an x, where $g(x) = 0$ and so $f(x) = f(x + \pi)$. For every meridian, we have a latitude value $l(x)$ for which the temperature works. Define now $h(x) = l(x) - l(x + \pi)$. This function is continuous. Start with the meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ have different signs. By the intermediate value theorem again, h has a root. There, both temperature and pressure agree with the antipode value.

Example: **Wobbly Table Theorem.** On an arbitrary floor, a square table can be turned so that it does not wobble any more.



Proof. The 4 legs ABCD are located on a square in a plane. Let x be the angle of the line AC with with a coordinate axes if we look from above. Given x, the table can be positioned **uniquely**: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. ($f(0) < 0$ is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.

Definition: Lets call $Df(x) = (f(x+h) - f(x))/h$ the h-**derivative** of f. We will study it more in the next lecture. You have in a homework already verified that $D\exp_h(x) = \exp_h(x)$.

Definition: Lets call a point p, where $Df(x) = 0$ a h-**critical point**. Lets call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a. Define similarly a **local minimum** as a point where $f(a) \leq f(x)$.

Example: The function $f(x) = x(x-h)(x-2h)$ has the derivative $Df(x) = 3x(x-h)$ as you have verified in the case $h = 1$ in the first lecture of this course in a worksheet. We will write $[x]^3 = x(x-h)(x-2h)$ and $[x]^2 = x(x-h)$. The computation just done tells that $D[x]^3 = 3[x]^2$. Since $[x]^2$ has exactly two roots $0, h$, the function $[x]^3$ has exactly 2 critical points.

Example: More generally for $[x]^{n+1} = x(x-h)(x-2h)\cdots(x-nh)$ we have $D[x]^{n+1} = (n+1)D[x]^n$. Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points. Keep the formula

$$\boxed{D[x]^n = n[x]^{n-1}}$$

in mind!

Example: You have verified that $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. Because this function has no roots and the derivative is the function itself, the function has no critical points.

$$\boxed{D\exp(x) = \exp(x)}$$

In our discrete setting we have a statement which later will be repeated when using actual critical points and when f has more regularity.

Fermat's maximum theorem If f is continuous and has a h-critical point a, then f has either a local maximum or local minimum inside the open interval $(a, a+h)$.

5.4. Look at the range of the function f restricted to $[a, a+h]$. It is a bounded interval $[c, d]$ because f is continuous. There exists especially a point u for which $f(u) = c$ and a point v for which $f(v) = d$. These points are different if f is not constant on $[a, a+h]$ and $b = f(a) = f(a+h)$ are in that interval. If $b = c$, then we have at least one local maximum v in $[a, a+h]$ if $b = d$ we have at least one local minimum u in $[a, a+h]$ if $c < b < d$, then there is both a local maximum v and a local minimum u in $[a, a+h]$. We will later argue with the intermediate value theorem.

Example: **Problem.** Verify that a quadratic polynomial has exactly one critical point. **Solution** $f(x) = ax^2 + bx + c$ with $a \neq 0$. Because the x^2 terms cancel, $f(x+h) - f(x)$ is a linear function $(2ah)x + (bh + ah^2)$. There is a root $x = (-b - ah)/(2a)$.

5.5. What we denote by "h-critical point" will in the limit $h \to 0$ be called "critical point" later in this course. While the h-critical point notion makes sense for all continuous functions. We will need more regularity to take the limit $h \to 0$. This limit $h \to 0$ will be considered in the future and functions for which the limit exists will be called **differentiable**. But we are not yet there.

Homework

Problem 5.1: Find the roots for $-x^5 - 3x^4 + 42x^3 + 62x^2 - 297x - 315$. You are told that all roots are integers.

Problem 5.2: Use the intermediate value theorem to verify that $f(x) = x^7 - 6x^6 + 8$ has at least two roots on $[-2, 2]$.

Problem 5.3: Motivated by the superball show: Beyoncé height is 169 cm. Shakira's height is 157 cm. Beyoncé was born September 4, 1981, Shakira was born February 2nd, 1977. Beyoncé is believed to be worth 400 million and Shakira's net worth is believed to be 300 million.
a) Can you argue that there was a moment when Shakira's height was exactly Beyoncé's height?
b) Can you argue that there was a moment when Shakira's age was exactly twice the age of Beyoncé?
c) Can you argue that there was a moment when Shakira's fortune was exactly half of Beyoncé's fortune?
d) Assume you live in New York. Show that if you drive the 190 miles from Boston to New York in 4 hours then there are at least two moments of time when you drive with exactly 40 miles per hours. The trip is not part of a larger trip. Your start is in Boston and end in New York.

Problem 5.4: Argue why there is a solution to
a) $5 - \sin(x) = x$, b) $\exp(7x) = x$, c) $\sin(x) = x^4$.
d) Why does the following argument not work:
The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.
e) Does the function $f(x) = x + \log|\log|x||$ have a root somewhere?

Problem 5.5: a) Find a concrete function f which has three local maxima and two local minima.
b) Let $h = 1/10$. Find a h-critical point for the function $f(x) = |x|$, that is a point for which $[f(x+h) - f(x)]/h = 0$.
c) Verify that for any $h > 0$, the function $f(x) = x^5$ has no h-critical point There is no x,l where $[f(x+h) - f(x)]/h = 0$ is possible.