INTRODUCTION TO CALCULUS

MATH 1A

Unit 4: Continuity

Lecture

4.1. Continuity is one of the most important concepts in mathematics:

> **Definition:** A **function** f is **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \to f(x_0)$ for $x \to x_0$. A function f is **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

4.2. In the interior (a, b), the limit needs to exist both from the right and from the left. At the boundary a of the interval, only the right limit needs to exist and at the point b, only the left limit. Intuitively, a function is continuous if you can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$.

4.3. Example. Any polynomial like x^3 or trig functions like $\cos(x), \sin(x), \exp(x)$ are continuous everywhere. Also the sum and product of such functions is continuous. For example, $x^5 + \sin(x^3 + x) - x\cos(x^7 + x^2)$ is continuous everywhere. We can also compose functions like $\exp(\sin(x))$ and still get a continuous function.

4.4. The function $f(x) = 1/x$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous everywhere. The discontinuity at $x = 0$ is also called a **pole**. One can draw a **vertical asymptote**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere. The function can be healed at 0 even so it was at first not defined at $x = 0$.

4.5. The function $f(x) = \log|x|$ is continuous for $x \neq 0$. It is not continuous at 0 because $f(x) \to -\infty$ for $|x| \to 0$. It might surprise you that $f(x) = (1 - x^2)/\log|x|$ is continuous everywhere. Yes, it is not defined at first at $x = 0$ as $\log|0| = -\infty$. It is also not defined at $x = 1$ or $x = -1$ at first because $\log(1) = 0$. But in both cases, we can heal. The value $f(0) = 0$ is easier to see, but filling in the value $f(1) = f(-1) = -2$ is less obvious. We will learn this later.

4.6. The co-secant function $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and more generally for any multiple of π. It has poles there because $\sin(x)$ is zero there so that we divide by zero at such points. The function $\cot(x) = \cos(x)/\sin(x)$ shares the same discontinuity points as $\csc(x)$.

4.7. The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just chose $x_n = 2/(4k+1)$ or $z_n = 2/(4k-1)$.

4.8. The **signum function** $f(x) = \text{sign}(x) = \left[\begin{array}{cc} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{array} \right.$ is not continuous at 0.

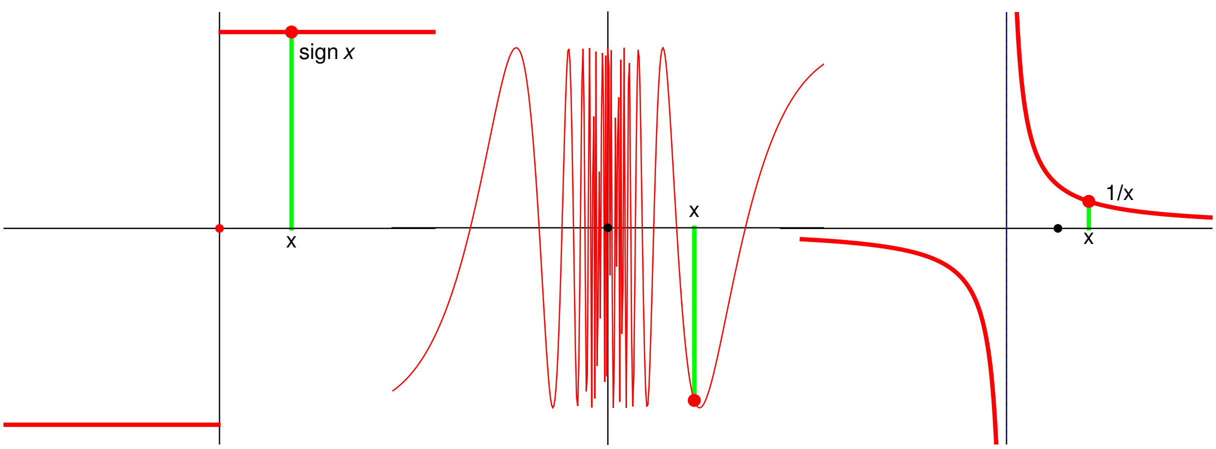
It is a prototype of a function with a **jump** discontinuity at 0.

Rules:
a) If f and g are continuous, then $f + g$ is continuous.
b) If f and g are continuous, then $f * g$ is continuous.
c) If f and g are continuous and if $g > 0$ then f/g is continuous.
d) If f and g are continuous, then $f \circ g$ is continuous.

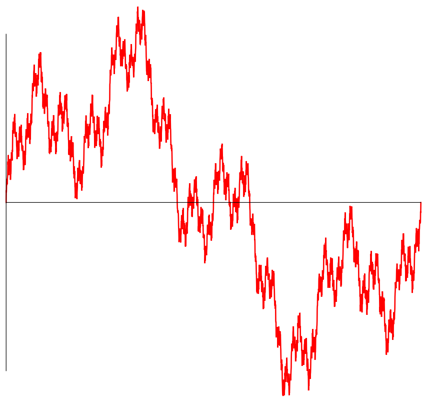
Example.
a) $f(x) = \sqrt{x^2 + 1}$ is continuous everywhere on the real line. b) $f(x) = \cos(x) + \sin(x)$ is continuous everywhere. c) $f(x) = \log(|x|)$ is continuous everywhere except at 0. d) $f(x) = \sin(\pi x)/\log|x^4|$ is continuous at $x = 0$. Is it continuous everywhere? Experiment.

Example: The function $f(x) = [\sin(x + h) - \sin(x)]/h$ is continuous for every parameter $h > 0$. We will see soon what happens when h becomes smaller and smaller and that the continuity will not deteriorate. Indeed, we will see that we get closer and closer to the cos function.

4.9. There are three major reasons, why a function is not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.

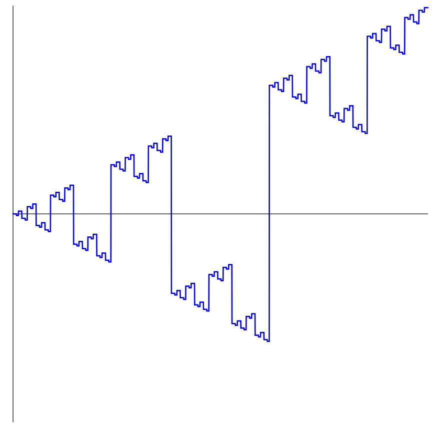


4.10. Why do we like continuity? We will see many reasons during this course but for now lets just say that:



A wild continuous function. This Weierstrass function is believed to be a fractal.

"Continuous functions can be pretty wild, but not too crazy."



A crazy discontinuous function. It is discontinuous at every point and known to be a fractal.

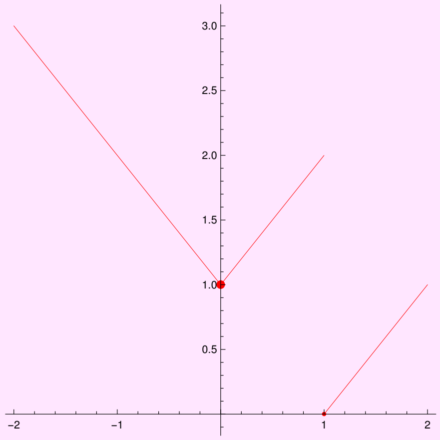
4.11. Continuity will be useful when finding maxima and minima. A continuous function on an interval $[a, b]$ has a maximum and minimum. We will see in the next hour that if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero. Being able to find solutions to equations $f(x) = 0$ is important and much more difficult, if f not continuous.

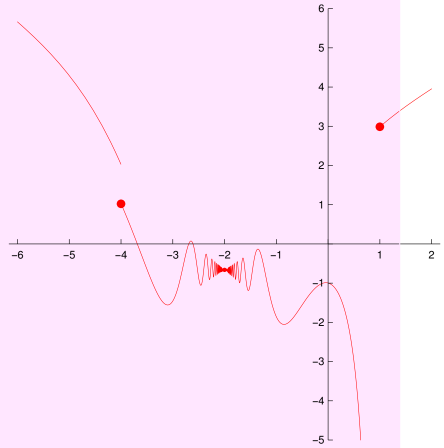
4.12. Problem Determine for each of the following functions, where discontinuities appear:
a) $f(x) = \log(|x^2 - 1|)$
b) $f(x) = \sin(\cos(\pi/x))$
c) $f(x) = \cot(x) + \tan(x) + x^4$
d) $f(x) = (x^2 + 2x + 1)/(x + 10 + (x - 1)^2/(x - 1)$
e) $f(x) = \frac{x^2-4x}{x}$

Homework

Problem 4.1: For the following functions, determine the points, where f is not continuous.
a) $\text{sinc}(x) + 1/\cos(x)$
b) $\sin(\tan(x))$
c) $f(x) = \cot(2 - x)$
d) $\text{sign}(x)/x$
e) $\frac{x^2+5x+x^4}{x-3}$
State which kind of discontinuity appears.

Problem 4.2: On which intervals are the following functions continuous?


a)


b)

Problem 4.3: Either do the following three problems a),b),c):
a) Construct a function which has a jump discontinuity and an escape to infinity.
b) Find a function which has an oscillatory discontinuity and an escape to infinity.
c) Find a function which has a jump discontinuity as well as an oscillatory discontinuity.
or shoot down the problem with one strike:
Find a function which has a jump discontinuity, a pole and an oscillatory discontinuity all at the same time.

Problem 4.4: Heal the following functions to make them continuous everywhere
a) $(x^3 - 8)/(x - 2)$
b) $(x^5 + x^3)/(x^2 + 3)$
c) $((\sin(x))^3 - \sin(x))/(\cos(x)\sin(x))$.
d) $(x^4 + 4x^3 + 6x^2 + 4x + 1)/(x^3 + 3x^2 + 3x + 1)$
e) $(x^{70} - 1)/(x^{10} - 1)$

Problem 4.5: Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, experiment by plotting the functions.
a) $\sin(\frac{1}{3+\sin(x)\cos(x)}) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 + \frac{7}{\exp(x)}$.
b) $\frac{2}{\log|x|} + x^7 - \cos(\sin(\cos(x))) - \exp(\log(\exp(x)))$