INTRODUCTION TO CALCULUS

MATH 1A

Unit 3: Limits

Lecture

3.1. The function $1/x$ is not defined everywhere. It blows up at $x = 0$, because we divide by zero. Sometimes however, functions can be **healed**. A silly example is $f(x) = x^2/x$ which is initially not defined at $x = 0$ because we divide by x but can be "saved" by noticing that $f(x) = x$ for all x different from 0. Functions often can be continued to "forbidden" places if we write the function differently. This can involves dividing out a common factor. Lets look at examples:

3.2. Example. The function $f(x) = (x^3 - 1)/(x - 1)$ is at first not defined at $x = 1$. But for x close to 1, nothing really bad happens. We can evaluate the function at points closer and closer to 1 and get closer and closer to 3. We say $\lim_{x \to 1} f(x) = 3$. Indeed, you might have noticed that $f(x) = x^2 + x + 1$ by factoring out $(x - 1)$. While initially not defined at $x = 1$, there is a natural value $b = 3$ we can assign for $f(3)$ so that the graph continues nicely through that point.

3.3. Definition. We write $x \to a$ to indicate that x approaches a. This approach can be from either side. A function $f(x)$ has a **limit** at a point a if there exists a unique b such that $f(x) \to b$ for $x \to a$. We write $\lim_{x \to a} f(x) = b$ if the limit exists and is the same from either side. In whatever way we approach it, we must get the same b. [1]

3.4. Example. The function $f(x) = \sin(x)/x$ is called $\text{sinc}(x)$. It is not defined at $x = 0$ at first. It appears naturally in geometry as a quotient between the length of a side of a right angle triangle and an arc length of a sector which contains it. Keep this function in mind. We will look at it later and prove that that the limit of $f(x)$ exists for $x \to 0$. It is so important that it is sometimes called the

Fundamental theorem of trigonometry. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

3.5. Example. The function $f(x) = x/|x|$ is 1 if $x > 0$ and -1 if $x < 0$. It is not defined at $x = 0$ and there is no way to assign a value b at $x = 0$ so that $\lim_{x \to 0} f(x) = b$. One can define $f(0) = 0$ we can call the function the *sign* **function**. It is defined everywhere but not continuous at 0.

[1] Technical: for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - b| < \epsilon$.

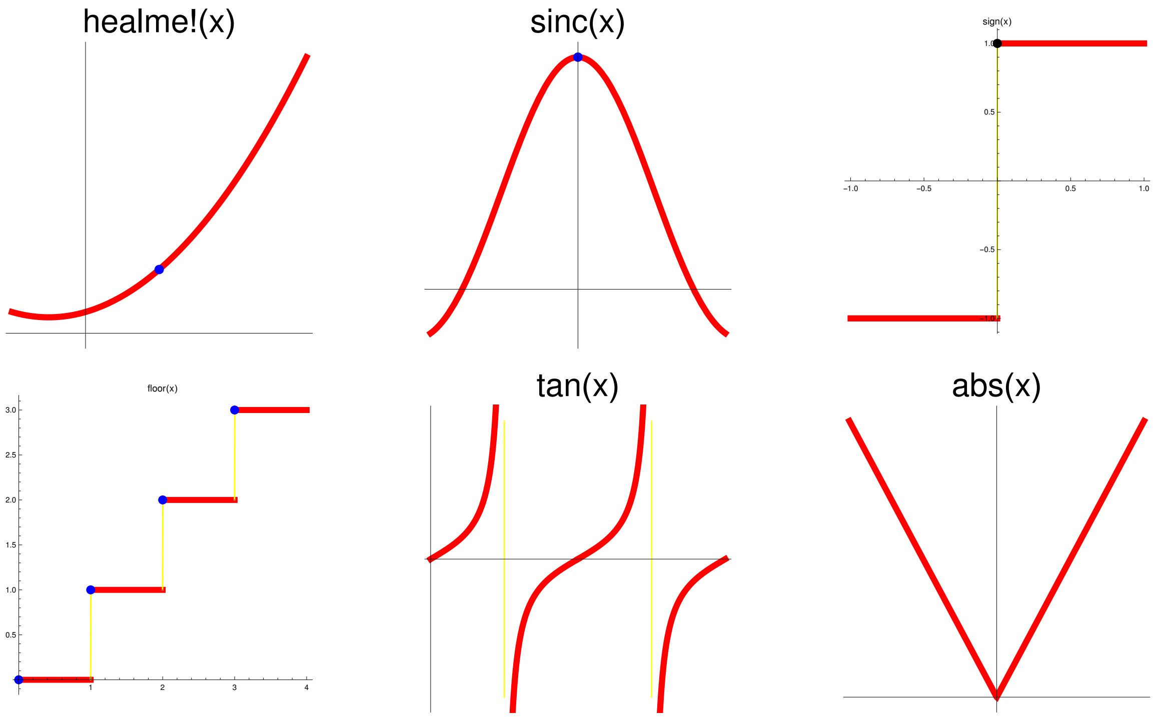


Figure: The graphs $f(x) = (x^3 - 1)/(x - 1)$, the sinc function $\text{sinc}(x) = \sin(x)/x$, the sign function $\text{sign}(x) = x/|x|$, the floor function $\text{floor}(x)$ giving the largest integer smaller or equal to x, the tan function and the **absolute value function** $\text{abs}(x) = |x|$.

3.6. Example. The quadratic function $f(x) = \cos(x^2)/(x^4 + 1)$ has the property that $f(x)$ approaches 1 if x approaches 0. To evaluate functions at 0, there was no need to take a limit because $x^4 + 1$ is never zero. The function is everywhere defined. Actually, most functions are nice in the sense that we do not have have to worry about limits at most points. In the overwhelming cases of real applications we only have to worry about limits when the function involves division by 0. For example $f(x) = (x^4 + x^2 + 1)/x$ needs to be investigated more carefully at $x = 0$. You see for example that for $x = 1/1000$, the function is slightly larger than 1000. We can simplify it to $x^3 + x + 1/x$ for $x \neq 0$. There is no limit $\lim_{x \to 0} f(x)$ because $1/x$ has no limit.

3.7. Example. Also, for sin and cos, the limit $\lim_{x \to a} f(x) = f(a)$ is defined. This extends to **trigonometric polynomials** like $\sin(3x) + \cos(5x)$. The function $\tan(x)$ however has no limit at $x = \pi/2$. No finite value b can be found so that $\tan(\pi/2 + h) \to b$ for $h \to 0$. This is due to the fact that $\cos(x)$ is zero at $\pi/2$.

3.8. Example. The **cube root** function $f(x) = x^{1/3}$ is defined for all x and even $x = 0$. For the square root function $f(x) = \sqrt{x}$ we have to be aware that for $x < -0$, it is not defined. The domain of the is function is the positive real axis.
Why do we worry about limits at all? One of the main reasons will is that we will soon define the derivative and integral using limits. An other reason is that one can use limits to define numbers like $\pi = 3.1415926\ldots$. In the next lecture, we also look at the important concept of continuity which refers to limits.

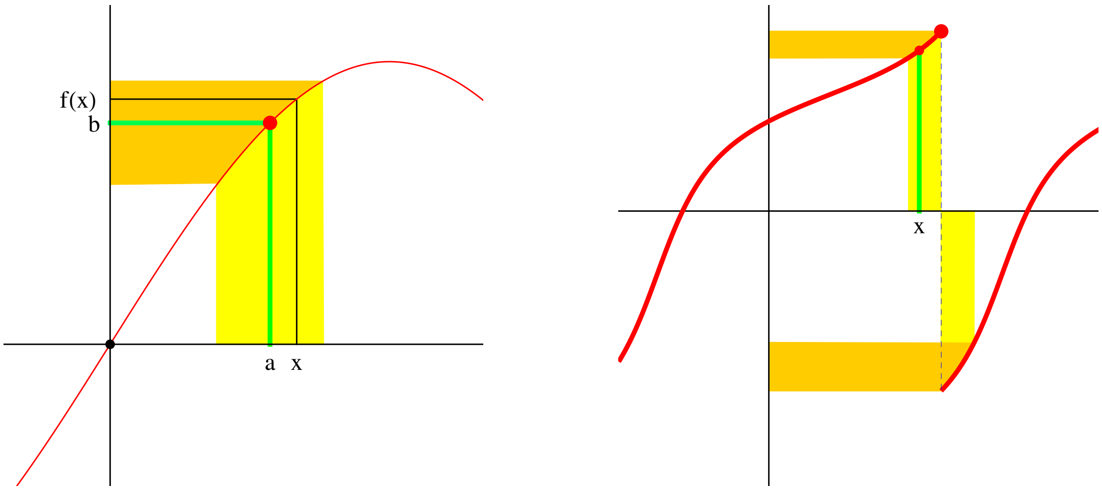


Figure: To the left we see a case, where the limit exists at $x = a$. If x approaches a then $f(x)$ approaches b. To the right we see the function $f(x) = \arctan(\tan(x) + 1)$, where arctan is the inverse of tan. The limit does not exist for $a = \pi/2$. If we approach a from the right, we get the limit $-\pi/2$. From the left, we get the limit $f(\pi/2) = \pi/2$. Note that f is not defined at $x = \pi/2$ because $\tan(x)$ becomes infinite there.

Example: Determine from the following functions whether the limits $\lim_{x\to 0} f(x)$ exist. If it does, find it.

a) $f(x) = \cos(x)/\cos(2x)$	b) $f(x) = \tan(x)/x$
c) $f(x) = (x^2 - x)/(x - 1)$	d) $f(x) = (x^4 - 1)/(x^2 - 1)$
e) $f(x) = (x + 1)/(x - 1)$	f) $f(x) = x/\sin(x)$
g) $f(x) = 5x/\sin(6x)$	h) $f(x) = \sin(x)/x^2$
i) $f(x) = \sin(x)/\sin(2x)$	j) $f(x) = \exp(x)/x$

3.9. The following properties hold for limits:

$\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ implies $\lim_{x\to a} f(x) + g(x) = b + c$.
$\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$ implies $\lim_{x\to a} f(x) \cdot g(x) = b \cdot c$.
$\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c \neq 0$ implies $\lim_{x\to a} f(x)/g(x) = b/c$.

3.10. This implies we can sum up and multiply or divide functions which have limits: Examples: Polynomials like $x^5 - 2x + 6$ or trig polynomials like $\sin(3x) + \cos(5x)$ have limits everywhere. Rational functions like $(x^2 - 1)/(x^2 + 1)$ have limits everywhere if the denominator has no roots. Functions like $\cos^2(x)\tan(x)/\sin(x)$ can be healed by simplification. Prototype functions like $\sin(x)/x$ have limits everywhere.

Homework

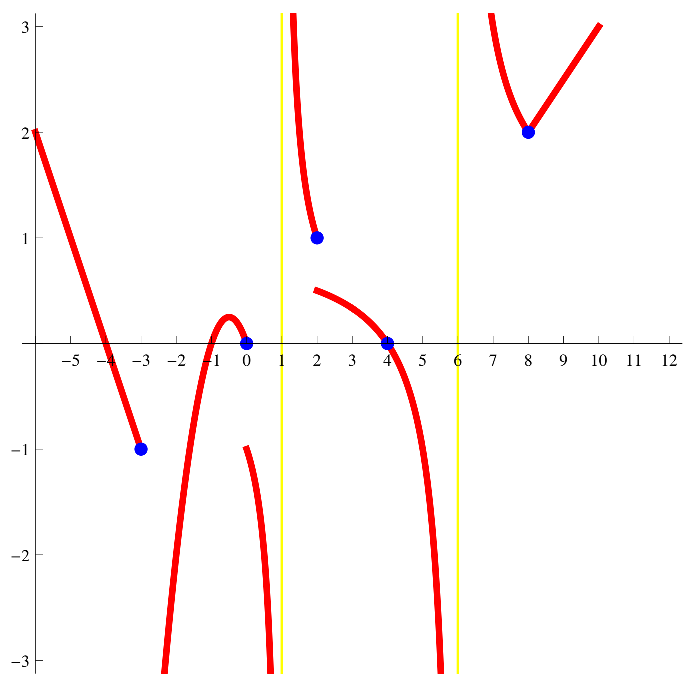
Problem 3.1: Find the limits $x \to 0$. You can use what we have established about $\mathrm{sinc}(x)$.
a) $f(x) = (x^6 - 1)/(x - 1)$, b) $f(x) = \sin(23x)/x$
c) $f(x) = \sin^2(9x)/x^2$, d) $f(x) = \sin(11x)/\sin(7x)$

Problem 3.2: a) Graph of the function

$$f(x) = \frac{(1-\cos(x))}{x^2} .$$

b) Where is the function f defined? Can you find the limit at the places, where it is not defined? Hint: **remember double angle formulas**
c) Verify that $f(x) = \exp_h(x) = (1+h)^{x/h}$ satisfies $[f(x+h) - f(x)]/h = f(x)$.
Remark. We define $e^x = \exp(x) = \lim_{h\to 0} \exp_h(x)$.

Problem 3.3: Find all points a at which the function given in the picture has no limits.



Problem 3.4: Find the limits for $x \to 0$:
a) $f(x) = (x^2 - 2x + 1)/(x-1)$,
b) $f(x) = \frac{\sin(x)}{x} \frac{2^x}{5^x}$.
c) $f(x) = 2^{(2^{2^x})}$,
d) $f(x) = \frac{\sin(\sin(x))}{\sin(x)}$.

Problem 3.5: We explore in this problem the limit of the function $f(x) = x^x$ if $x \to 0$. Write a short research paragraph about it. It should involve some experiments and cases. Can we find a limit in general? Take a calculator or use Wolfram α and experiment. What do you see when $x \to 0$? Can you find an explanation for your experiments? Discuss!