INTRODUCTION TO CALCULUS

MATH 1A

Unit 2: Functions

Lecture

2.1. A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^3 - 2x$ for example assigns to the number $x = 2$ the value $2^3 - 4 = 4$. A function is given with a **domain** A, the points where f is defined and a **codomain** B a set of numbers which f can reach. Usually, functions are defined everywhere, like the function $f(x) = x^2 - 2x$. If this is the case, we often do not mention the domain or assume that the domain is the place where the function is defined. A function $g(x) = 1/x$ for example can not be evaluated at 0 so that the domain must exclude the point 0. The **inverse** of a function f is a function g such that $g(f(x)) = x$. The function $g(x) = \sqrt{x}$ is the inverse of the function $f(x) = x^2$ on the positive axes $[0, \infty)$.

2.2. Here are a few examples of functions. We will look at them in more detail during the lecture. Very important are polynomials, trigonometric functions, the exponential and logarithmic function. You won't find the h-exponential in any textbook. We will have a bit of fun with them later. If you want take them in the case $h = 1$ and later in the case $h \to 0$, where it is the usual exponential $\exp(x)$. They are the exponentials and logarithms in "quantum calculus" and will in the limit $h \to 0$ become the regular exponential and logarithm functions.

constant	$f(x) = 1$	power	$f(x) = 2^x$
identity	$f(x) = x$	exponential	$f(x) = e^x = \exp(x)$
linear	$f(x) = 3x + 1$	logarithm	$f(x) = \log(x) = \ln(x)$
quadratic	$f(x) = x^2$	absolute value	$f(x) = \|x\|$
cosine	$f(x) = \cos(x)$	devil comb	$f(x) = \sin(1/x)$
sine	$f(x) = \sin(x)$	bell function	$f(x) = e^{-x^2}$
exponentials	$f(x) = \exp_h(x) = (1+h)^{x/h}$	Agnesi	$f(x) = \frac{1}{1+x^2}$
logarithms	$f(x) = \exp_h^{-1}$	sinc	$\sin(x)/x$

We can build new functions by:

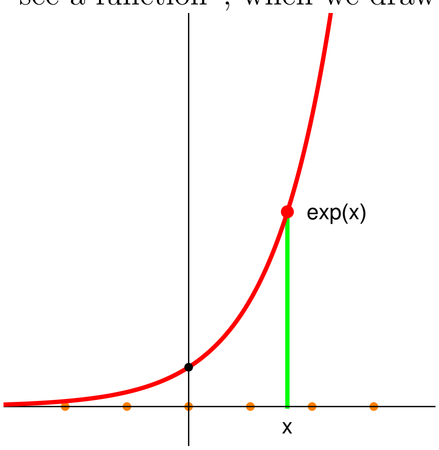
addition	$f(x) + g(x)$
scaling	$2f(x)$
translating	$f(x+1)$
compose	$f(g(x))$
invert	$f^{-1}(x)$

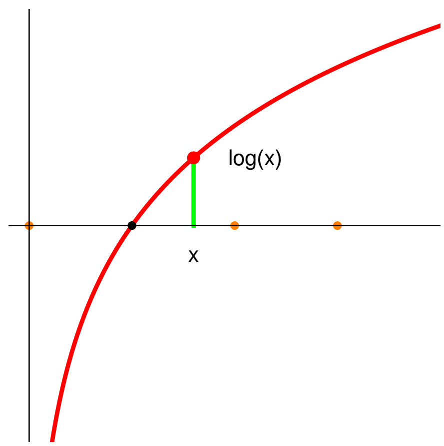
Here are important functions:

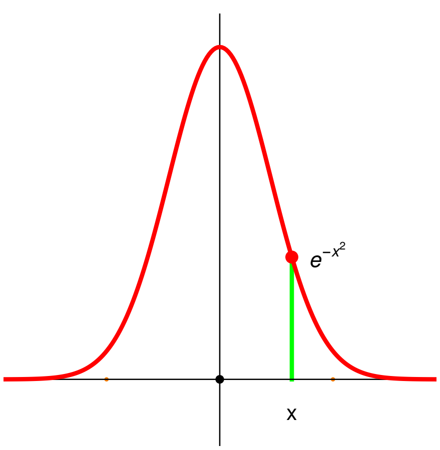
polynomials	$x^2 + 3x + 5$
rational functions	$(x+1)/(x^4+1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin(x), \arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

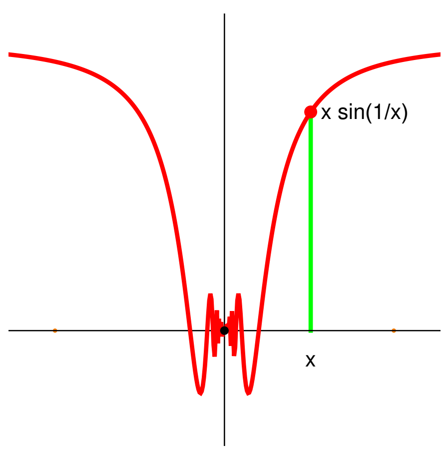
2.3. We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important. For some functions, we need to restrict the domain, where the function is defined. For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number x on which we evaluate the function is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

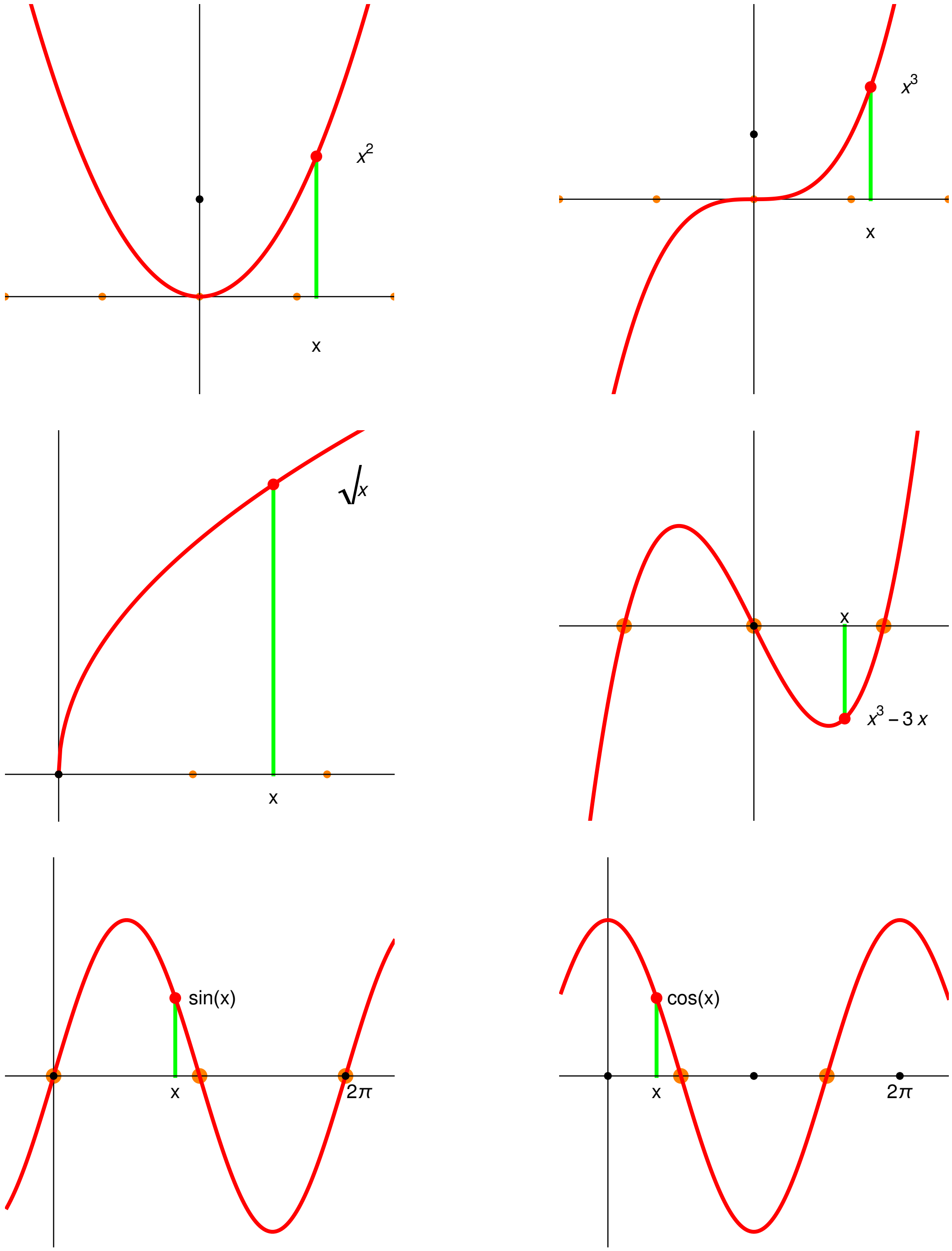
2.4. The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\ \}$ in the plane, where x runs over the domain A of f. Graphs allow us to **visualize** functions. We can "see a function", when we draw the graph.











2.5. Definition: A function $f : A \to B$ is **invertible** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. The function g is the **inverse** of f. Example: $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0, \infty)$ to $B = [0, \infty)$. You can check with the **horizontal line test** whether an inverse exists: draw the box with base A and side B, then every horizontal line should intersect the graph exactly once.

Homework

Here is the homework for this section.

Problem 2.1: Draw the function $f(x) = x^3 \sin(4x)$ on the interval $[-5, 5]$. Its graph goes through the origin $(0, 0)$. You can use technology.
a) A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
b) A function is called **even** if $f(-x) = f(x)$. Is f even?
c) What happens in general if a function f is both even and odd?

Problem 2.2: Determine from the following functions whether they are invertible. and write down the inverse if they are
a) $f(x) = x^7 - 1$ from $A = \mathbb{R}$ to $B = \mathbb{R}$
b) $f(x) = \cos(x^3)$ from $A = [0, \pi/2]$ to $B = [0, 1]$
c) $f(x) = \sin(x)$ from $A = [0, \pi]$ to $B = [0, 1]$
d) $f(x) = \tan(x)$ from $A = (-\pi/2, \pi/2)$ to $B = \mathbb{R}$.
e) $f(x) = 1/(1 + x^2)$ from $A = [0, \infty)$ to $B = (0, 1]$.

Problem 2.3: a) Draw the graphs of $\exp_1(x) = 2^x, \exp_{1/10}(x) = (1 + 1/10)^{10x}$ and $\exp(x)$.
b) Draw the graphs the inverse of these functions.
You are welcome to use technology for a). For b), just "flip the graph" at the line $x = y$.

Problem 2.4: Try to plot the function $\exp(\exp(\exp(x)))$ on $[0, 1]$. This is a fine function but computer programs do not plot always graph. Describe what you see when the machine plots the function.

Problem 2.5: A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Find at least one root for each of the following functions.
a) $f(x) = \cos(x)$
b) $f(x) = 4\exp(-x^4)$
c) $f(x) = x^5 - x^3$
d) $f(x) = \log(x) = \ln(x)$
e) $f(x) = \sin(x) - 1$
f) $f(x) = \csc(x) = 1/\sin(x)$

(*) Here is how you to plot a function:

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http://www.wolframalpha.com/input/?i=Plot+sin(x)
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Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2020