Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

5/7/2020: Final Practice B

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ 20: False. Sign your paper.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T **F** $\frac{d}{dx} e^{e^x} = e^x$.

Solution:
Use the chain rule.

2) **T** F A function f which is concave down at 0 satisfies $f''(0) \leq 0$.

Solution:
Yes, this can even be taken as a definition of concavity

3) T **F** The integral $\int_{1/2}^{1} \log(x)\ dx$ is positive. Here $\log(x) = \ln(x)$ is the natural log.

Solution:
The integrand is always negative.

4) **T** F The function $x + \sin(\cos(\sin(x)))$ has a root in the interval $(-10, 10)$.

Solution:
Use the intermediate value theorem.

5) **T** F The function $x(1-x) + \sin(\sin(x(1-x)))$ has a maximum or minimum inside the interval $(0, 1)$.

Solution:
Use Rolle's theorem

6) T **F** The derivative of $1/(1+x^2)$ is equal to $\arctan(x)$.

Solution:
It is the other way round.

7) T **F** The limit of $\sin^{100}(x)/x^{100}$ for $x \to 0$ exists and is equal to 100.

Solution:
It is equal to 1.

8) T **F** The function $f(x) = (1 - e^x)/\sin(x)$ has the limit 1 as x goes to zero.

Solution:
Use Hopital's rule

9) **T** F The frequency of the sound $\sin(10000x)$ is higher than the frequency of $\sin(\sin(3000x))$.

Solution:
Yes, about 3 times larger. The frequency is 10'000 versus 3000.

10) **T** F The function $f(x) = \sin(x^2)$ has a local minimum at $x = 0$

Solution:
The function is positive near 0 but equal to zero at 0.

11) **T** F The function $f(x) = (x^5 - 1)/(x - 1)$ has a limit for $x \to 5$.

Solution:
Use Hopital's rule, or heal the function.

12) **T** F The average cost $g(x) = F(x)/x$ of an entity is extremal at x for which $f(x) = g(x)$. Here, $f(x)$ denotes the marginal cost and $F(x)$ the total cost.

Solution:
This is the strawberry theorem.

13) T **F** The mean of a probability density function is defined as $\int f(x)\, dx$.

Solution:
No, it is equal to 1.

14) T **F** The differentiation rule $(f(x)^{g(x)})' = (f'(x))^{g(x)} g'(x)$ holds for all differentiable functions f, g.

Solution:
No there is no such rule.

15) **T** F $\sin(5\pi/6) = 1/2$.

Solution:
Yes, it is equal to $\sin(\pi/6)$.

16) **T** F Hôpital's rule assures that $\sin(10x)/\tan(10x)$ has a limit as $x \to 0$.

Solution:
Yes, the limit is 1

17) T **F** A Newton step for the function f is $T(x) = x - \frac{f'(x)}{f(x)}$.

Solution:
Wrong. The derivative is in the denominator.

18) T **F** A minimum x of a function f is called a catastrophe if $f'''(x) < 0$.

Solution:
Nonsense, where did he get this from?

19) | T | F | The fundamental theorem of calculus implies $\int_{-1}^{1} g'(x)\ dx = g(1) - g(-1)$ for all differentiable functions g.

Solution:
Yes, even if the function is called g.

20) | T | F | If f is a differentiable function for which $f'(x) = 0$ everywhere, then f is constant.

Solution:
Yes, integrating shows $f = c$.

Problem 2) Matching problem (10 points) No justifications needed

a) (2 points) One of three statements A)-C) is not the part of the fundamental theorem of calculus. Which one?

A)	$\int_0^x f'(t)\ dt = f(x) - f(0)$
B)	$\frac{d}{dx} \int_0^x f(t)\ dt = f(x)$
C)	$\int_a^b f(x)\ dx = f(b) - f(a)$

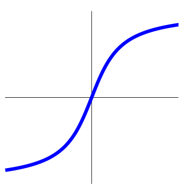
b) (3 points) Biorythms can be fascinating for small kids, giving them a first exposure to trig functions and basic arithmetic. The "theory" tells that there are three functions $p(x) = \sin(2\pi x/23)$ (Physical) $e(x) = \sin(2\pi x/28)$ (Emotional) and $i(x) = \sin(2\pi x/33)$ (Intellectual), where x is the number of days since your birth. Assume **Tuck**, the pig you know from the practice exams, is born on October 10, 2005. Today, on May 11, 2014, it is 2670 days old. Its biorythm is $E = 0.7818, P = -0.299, I = -0.5406$. It is a happy fellow, tired, but feeling a bit out of spirit, like the proctor of this exam feels right now. Which of the following statements are true?

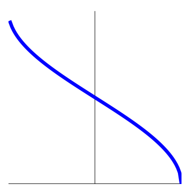
Check if true	
	i) One day old Tuck had positive emotion, intellect and physical strength.
	ii) Among all cycles, the physical cycle takes the longest to repeat.
	iii) Comparing with all cycles, the physical increases fastest at birth.

c) (4 points) Name the statements:

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ is called the	.
Rule $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$ is called	.
$\int_0^x f'(t)\, dt = f(x) - f(0)$ is called	.
The PDF $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is called the	

d) (1 point) Which of the following graphs belongs to the function $f(x) = \arctan(x)$?


1)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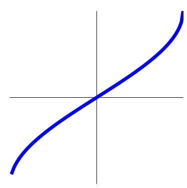
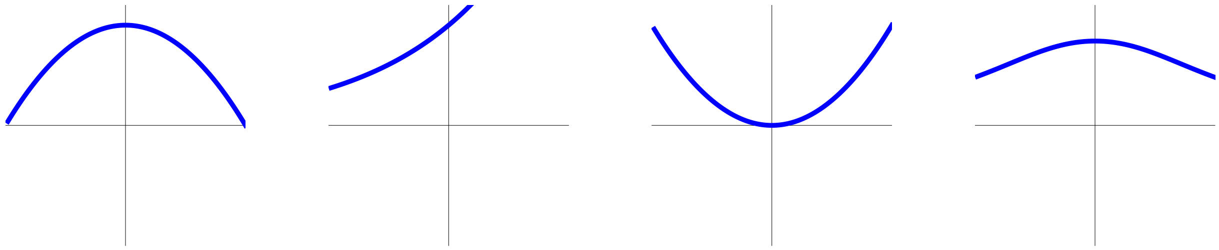

3)

Solution:
a) C
b) i) and iii)
c) Fundamental Theorem of trigonometry, Hopital's rule, Fundamental theorem of calculus, Normal distribution.
d) The first picture 1).

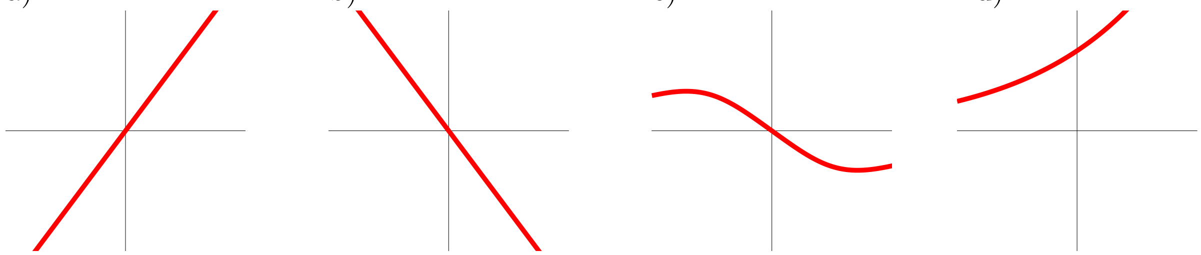
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Match the functions (a-d) (top row) with their derivatives (1-4) (middle row) and second derivatives (A-D) (last row).

Function a)-d)	Fill in 1)-4)	Fill in A)-D)
graph a)		
graph b)		
graph c)		
graph d)		



a) b) c) d)



1) 2) 3) 4)

A) B) C) D)

b) (4 points) Match the following integrals with the areas in the figures:

Integral	Enter 1-4
$\int_{-\pi}^{\pi} x \sin(x)\ dx.$	
$\int_{-\pi}^{\pi} \exp(-x^2)\ dx.$	

Integral	Enter 1-4
$\int_{-\pi}^{\pi} \pi + x\ dx.$	
$\int_{-\pi}^{\pi} 1 - \sin(x^3/\pi^3)\ dx.$	

1) 2) 3) 4)

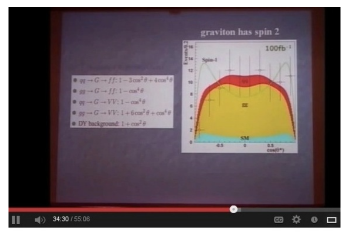
c) (2 points) Name two different numerical integration methods. We have seen at least four.

Your first method	
Your second method	

Solution:
a) 2D,4A,1C,3B
b) The middle two graphs are even functions and belong to the left box.
$\int_{-\pi}^{\pi} x \sin(x)\, dx$. matches 2
$\int_{-\pi}^{\pi} \exp(-x^2)\, dx$. matches 3
$\int_{-\pi}^{\pi} \pi + x\, dx$. matches 1
$\int_{-\pi}^{\pi} 1 - \sin(x^3/\pi^3)\, dx$. matches 4 c) We have seen Simpson, Trapezoid, MonteCarlo

Problem 4) Area computation (10 points)

A slide in a lecture of Harvard physicist **Lisa Randall** shows the area between two functions. Lisa is known for her theory of "branes" which can explain why gravity is so much weaker than electromagnetism. Assist Lisa and write down the formula for the area between the graphs of $1 - \cos^2(x)$ and $1 - \cos^4(x)$, where $0 \leq x \leq \pi$. Find the area.



Lisa Randall - Extra Dimensional Particle Resonances at the LHC (1/3)

Hint. Lisa already knows the identity

$$\cos^2(x) - \cos^4(x) = \cos^2(x)(1 - \cos^2(x)) = \cos^2(x)\sin^2(x)\ .$$

Solution:
By the hint, we are led to the integral

$$\int_0^{\pi} \cos^2(x)\sin^2(x)\, dx = \frac{1}{4}\int_0^{\pi} \sin^2(2x)\, dx = \frac{1}{8}\int_0^{\pi} (1 - \cos(4x))\, dx = \frac{\pi}{8}\ .$$

The answer is $\boxed{\pi/8}$. The key were double angle formulas. [An other possibility was to use $\cos^2(x) = (1 + \cos(2x))/2, \sin^2(x) = (1 - \cos(2x))/2$ and then use the double angle formula $\sin(4x) = 2\cos(2x)\sin(2x)$ later.]

Problem 5) Volume computation (10 points)

Find the volume of the solid of revolution for which the radius at height z is

$$r(z) = \sqrt{z \log(z)}$$

and for which z is between 1 and 2. Here, log is the natural log. Naturalmente!

Solution:
The integral $\pi \int z \log(z)\ dz$ can be done using integration by parts by differentiating log first (LIATE,LIPTE). We have $\pi(z^2 \log(z)/2 - \int z/2\ dz) = \pi(\log(z)/2 - 1/4)z^2$. The definite integral is $\boxed{\pi(\log(4) - 3/4)}$.

Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{7}{x^{3/4}}\ dx\ .$$

b) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{13}{x^{5/4}}\ dx\ .$$

Solution:
a)

$$7\int_1^\infty \frac{1}{x^{3/4}}\ dx = 28x^{1/4}|_1^\infty = \infty\ .$$

b)

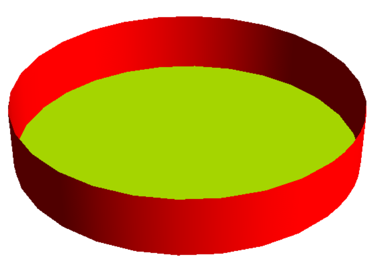
$$13\int_1^\infty \frac{1}{x^{5/4}}\ dx = -52x^{-1/4}|_1^\infty = 52\ .$$

Problem 7) Extrema (10 points)

A **candle holder** of height y and radius x is made of aluminum. Its total surface area is $2\pi xy + \pi x^2 = \pi$ (implying $y = 1/(2x) - x/2$). Find x for which the volume

$$f(x) = x^2 y(x)$$

is maximal.



Solution:
Substituting $y(x)$ in gives the function $f(x) = x/2 - x^3/2$ which has the derivative $f'(x) = 1/2 - 3x^2/2$. It is zero for $\boxed{x = 1/\sqrt{3}}$. The second derivative $-3x$ is negative there so that this is a maximum.

Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (x+5)^3 \sin(x-4)\, dx \ .$$

b) (5 point) Find the indefinite integral

$$\int e^x \cos(2x)\, dx \ .$$

Don't get dizzy when riding this one.

Solution:
a) Use the Tic-Tac-Toe integration method:

$(x+5)^3$	$\sin(x-4)$	
$3(x+5)^2$	$-\cos(x-4)$	$\oplus$
$6(x+5)^1$	$-\sin(x-4)$	$\ominus$
6	$\cos(x-4)$	$\oplus$
0	$\sin(x-4)$	$\ominus$

We can read off the answer $-(x-5)^3\cos(x-4) + 3(x+5)^2\sin(x-4) + 6(x+5)\cos(x-4) - 6\sin(x-4) + C$.
b) We use the merry go round by using integration by parts twice calling the integral I. We have

$$I = \cos(2x)e^x + \int 2\sin(2x)e^x\, dx = \cos(2x) + 2\sin(2x) - 4I \ .$$

Solving for I gives $\boxed{(\cos(x) + 2\sin(2x))e^x/5 + C}$.

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int \log(x^3)x^2\, dx$.

b) (4 points) Solve the integral $\int x\cos(x^2)\exp(\sin(x^2))\,dx$.

c) (3 points) Find the integral $\int \sin(\exp(x))\exp(x)\,dx$.

Solution:
These are all standard substitution problems:
a) Substitute $u = x^3$ to get $(x^3\log(x^3) - x^3)/3 + C$
b) Substitute $u = \sin(x^2)$ to get $\exp(\sin(x^2)) + C$.
c) Substitute $u = \exp(x)$ to get $-\cos(\exp(x)) + C$.

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-2)(x-3)(x-4)}\,dx .$$

(Evaluate the absolute values $\log|\cdot|$ in your answer. The improper integrals exist as a Cauchy principal value).

b) (5 points) Find the indefinite integral

$$\int \frac{1}{x(x-1)(x+1)(x-2)}\,dx .$$

Solution:
a) We use the hopital method to find the constants $A = 1/2, B = -1, C = 1/2$ in

$$\frac{1}{(x-2)(x-3)(x-4)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} + \frac{C}{(x-4)} .$$

The answer is $\log(x-2)/2 - \log(x-3) + \log(x-4)/2$. The definite integral is $\boxed{\text{zero}}$.
b) Again use Hopital to get $A = 1/2, B = -1/2, C = -1/6, D = 1/6$ in

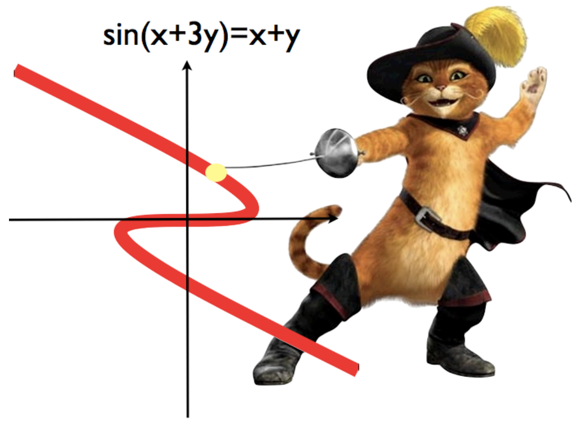
$$\frac{1}{x(x-1)(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2} .$$

The answer is

$$\log(x)/2 - \log(x-1)/2 - \log(x+1)/6 + \log(x-2)/6 + C .$$

Problem 11) Chain rule. (10 points)

a) Find the derivative of $f(g(x))$, where $f(x) = \sin(\pi x)$ and $g(x) = x^4 + 3x$. b) Let us look at a specific point x. While x is unknown, you know $g(x) = 4$ and $g'(x) = 7$. What is $\frac{d}{dx} f(g(x))$ at this point?



Solution:
a) The derivative is $\pi \cos(\pi(x^4 + 3x))(4x^3)$.
b) It is $\cos(4\pi)7 = 7$.

Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = x \log(x) + \frac{1}{1+x^2}$.

b) (3 points) $f(x) = \frac{2x}{x^2+1} + \frac{1}{x^2-4}$.

c) (2 points) $f(x) = \sqrt{16 - x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{x \log(x)}$.

Solution:
1) The first part appeared in the volume problem before. We have a) $x^2 \log(x)/2 - x^2/4 + \arctan(x) + C$.
b) $\log(x^2+1) - \log(x-2)/4 + \log(x+2)/4 + C$.
c) $\arcsin(x/4) + (1 - \cos(2\arcsin(x/4)))/4 + C$.
d) $x\log(x) - x + \log(\log(x)) + C$.

Problem 13) Applications (10 points)

a) (2 points) Find the CDF $\int_0^x f(t)\,dt$ for the PDF which is $f(x) = \exp(-x/3)/3$ for $x \geq 0$ and 0 for $x < 0$.

b) (2 points) Perform a single Newton step for the function $f(x) = \sin(x)$ starting at $x = \pi/3$.

c) (2 points) Check whether the function $f(x) = 1/(2x^2)$ on the real line $(-\infty, \infty)$ is a probability density function.

d) (2 points) A rower produces the power $P(t)$ is $\sin^2(10t)$. Find the energy $\int_0^{2\pi} P(t)\,dt$ when rowing starting at time $t = 0$ and ending at $t = 2\pi$.

e) (2 points) What is the frequency of the Midi number 10?

Solution:
a) The antiderivative is $\boxed{(1 - e^{-x/3}}$.
b) $\pi/3 - \tan(\pi/3) = \boxed{\pi/3 - \sqrt{3}}$.
c) The improper integral does not exist. It is **not** a probability density function.
d) Integrate $\int_0^{2\pi} \sin^2(10t)\,dt = \int_0^{2\pi} (1 - \cos(20t))/2\,dx = x/2 - \sin(20t)/40|_0^{2\pi} = \boxed{\pi}$.
e) $f = 440e^{(10-69)/12}$.