Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

5/7/2020: Final Practice E

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ 20: False. Sign your paper.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F If a function $f(x)$ has a critical point 0 and $f''(0) = 0$ then 0 is neither a maximum nor minimum.

2) T F If $f' = g$ then $\int_0^x g(x) = f(x)$.

3) T F The function $f(x) = 1/x$ has the derivative $\log(x)$.

4) T F The function $f(x) = \arctan(x)$ has the derivative $1/\cos^2(x)$.

5) T F The fundamental theorem of calculus implies that $\int_a^b f'(x)\ dx = f(b) - f(a)$.

6) T F $\lim_{x\to 8} 1/(x-8) = \infty$ implies $\lim_{x\to 3} 1/(x-3) = \omega$.

7) T F A continuous function which satisfies $\lim_{x\to-\infty} f(x) = 3$ and $\lim_{x\to\infty} f(x) = 5$ has a root.

8) T F The function $f(x) = (x^7 - 1)/(x - 1)$ has a limit at $x = 1$.

9) T F If $f_c(x)$ is an even function with parameter c and $f'(0) = 0$ and for $c < 3$ the function is concave up at $x = 0$ and for $c > 3$ the function is concave down at $x = 0$, then $c = 3$ is a catastrophe.

10) T F The function $f(x) = +\sqrt{x^2}$ has a continuous derivative 1 everywhere.

11) T F A rower rows on the Charles river leaving at 5 PM at the Harvard boat house and returning at 6 PM. If $f(t)$ is the distance of the rower at time t to the boat house, then there is a point where $f'(t) = 0$.

12) T F A global maximum of a function $f(x)$ on the interval $[0, 1]$ is a critical point.

13) T F A continuous function on the interval $[2, 3]$ has a global maximum and global minimum.

14) T F The intermediate value theorem assures that if f is continuous on $[a, b]$ then there is a root of f in (a, b).

15) T F On an arbitrary floor, a square table can be turned so that it does not wobble any more.

16) T F The derivative of $\log(x)$ is $1/x$.

17) T F If f is the marginal cost and $F = \int_0^x f(x)\ dx$ the total cost and $g(x) = F(x)/x$ the average cost, then points where $f = g$ are called "break even points".

18) T F At a function party, Log talks to Tan and the couple Sin and Cos, when she sees her friend Exp alone in a corner. Log: "What's wrong?" Exp: "I feel so lonely!" Log: "Go integrate yourself!" Exp sobbs: "Won't change anything." Log: "You are so right".

19) T F If a car's position at time t is $f(t) = t^3 - t$, then its acceleration at $t = 1$ is 6.

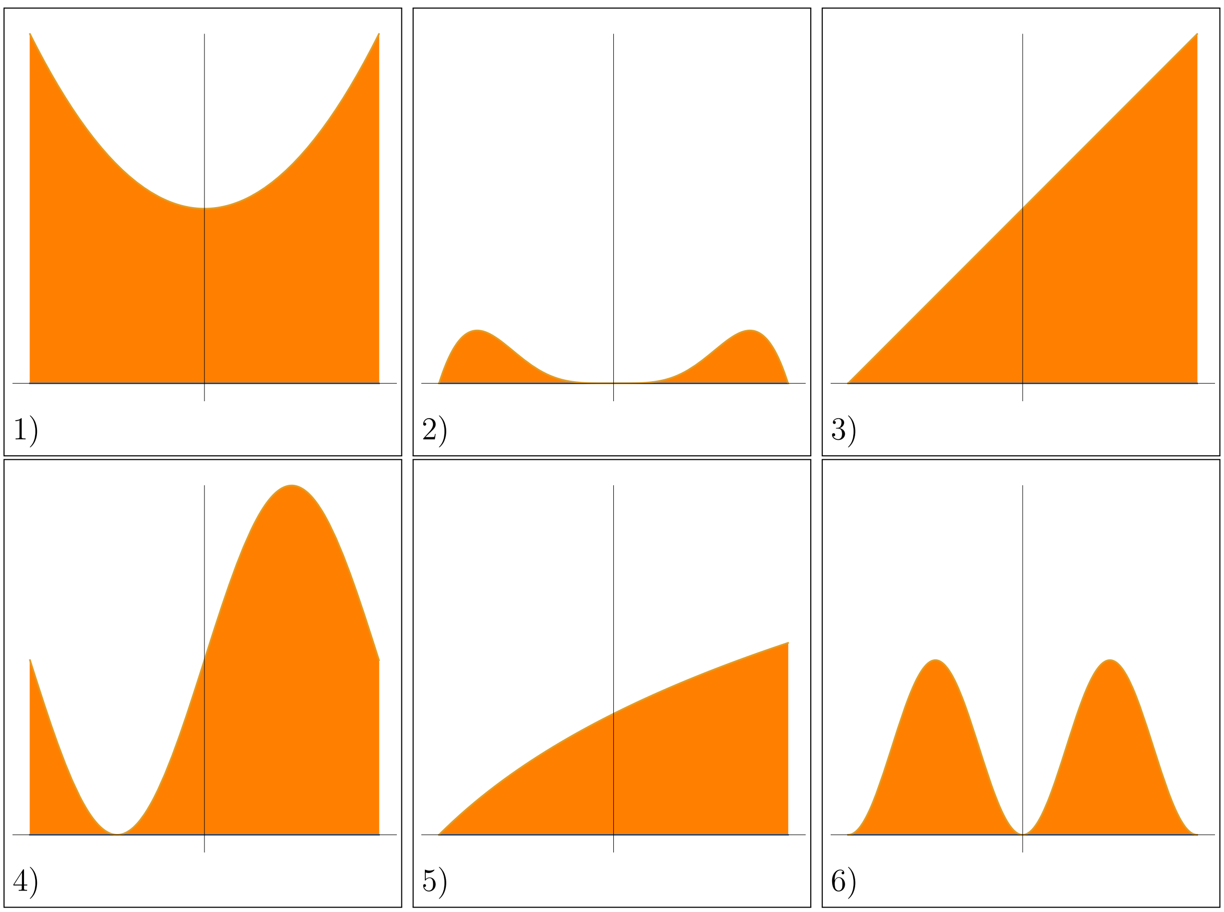
20) [T] [F] For trig substitution, the identities $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$, $\cos(x) = \frac{1-u^2}{1+u^2}$ are useful.

Problem 2) Matching problem (10 points) No justifications are needed.

a) Match the following integrals with the graphs and (possibly signed) areas.

Integral	Enter 1-6
$\int_{-1}^{1} \sin(\pi x)x^3 \, dx$.	
$\int_{-1}^{1} \log(x+2) \, dx$.	
$\int_{-1}^{1} x+1 \, dx$.	

Integral	Enter 1-6
$\int_{-1}^{1} (1+\sin(\pi x)) \, dx$.	
$\int_{-1}^{1} \sin^2(x) \, dx$.	
$\int_{-1}^{1} x^2+1 \, dx$.	



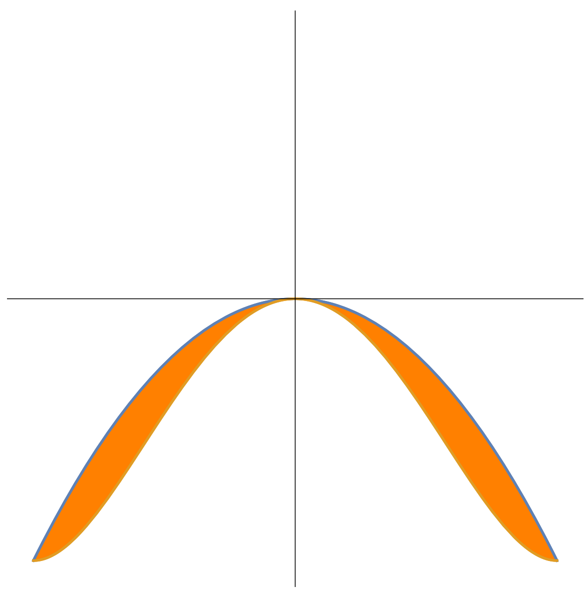
Problem 3) Matching problem (10 points) No justifications are needed.

Determine from each of the following functions, whether discontinuities appears at $x = 0$ and if, which of the three type of discontinuities it is at 0.

Function	Jump discontinuity	Infinity	Oscillation	No discontinuity
$f(x) = \log(\|x\|^5)$				
$f(x) = \cos(5/x))$				
$f(x) = \cot(1/x)$				
$f(x) = \sin(x^2)/x^3$				
$f(x) = \arctan(\tan(x - \pi/2))$				
$f(x) = 1/\tan(x)$				
$f(x) = 1/\sin(x)$				
$f(x) = 1/\sin(1/x)$				
$f(x) = \sin(\exp(x))/\cos(x)$				
$f(x) = 1/\log\|x\|$				

Problem 4) Area computation (10 points)

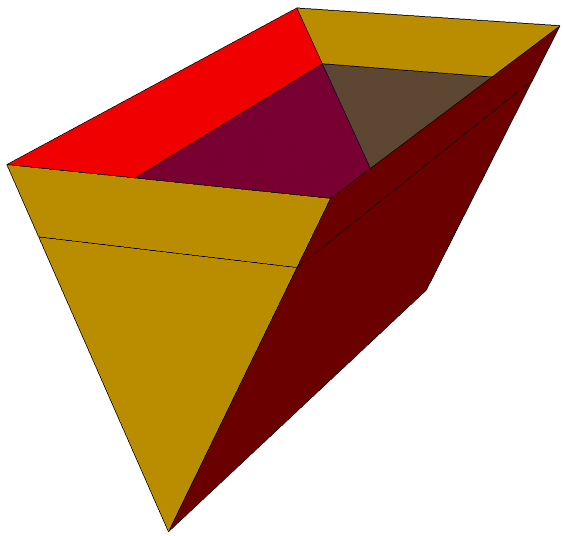
Find the area of the region enclosed by the graphs of the function $f(x) = x^4 - 2x^2$ and the function $g(x) = -x^2$.



Problem 5) Volume computation (10 points)

A farmer builds a bath tub for his warthog "Tuk". The bath has triangular shape of length 10 for which the width is $2z$ at height z. so that when filled with height z the surface area of the water is $20z$. If the bath has height 1, what is its volume?

P.S. Don't ask how comfortable it is to soak in a bath tub with that geometry. The answer most likely would be "Noink Muink".



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (3 points) $\int_1^2 \sqrt{x} + x^2 - 1/\sqrt{x} + 1/x \; dx$.

b) (3 points) $\int_1^2 2x\sqrt{x^2-1} \; dx$

c) (4 points) $\int_1^2 2/(5x-1) \; dx$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (3 points) $\int \frac{3}{1+x^2} + x^2 \; dx$

b) (3 points) $\int \frac{\tan^2(x)}{\cos^2(x)}\, dx$

c) (4 points) $\int \log(5x)\, dx$.

Problem 8) Chain rule (10 points)

A juice container of volume $V = \pi r^2 h$ changes radius r but keeps the height $h = 2$ fixed. Liquid leaves at a constant rate $V'(t) = -1$. At which rate does the radius of the bag shrink when $r = 1/2$? Differentiate $V(t) = \pi r(t)^2 h$ for the unknown function $r(t)$ and solve for $r'(t)$, then evaluate for $r = 1/2$.

Problem 9) Global extrema (10 points)

We build a chocolate box which has 4 cubical containers of dimension $x \times x \times h$. The total material is $f(x, h) = 4x^2 + 12xh$ and the volume is $4x^2h$. Assume the volume is 4, what geometry produces the minimal cost?

Problem 10) Integration techniques (10 points)

Which integration technique works? It is enough to get the right technique and give the first step, not do the actual integration:

a) (2 points) $\int (x^2 + x + 1)\sin(x)\, dx$.

b) (2 points) $\int x/(1 + x^2)\, dx$.

c) (2 points) $\int \sqrt{4 - x^2}\, dx$.

d) (2 points) $\int \sin(\log(x))/x$.

e) (2 points) $\int \frac{1}{(x-6)(x-7)}\, dx$.

Problem 11) Hopital's rule (10 points)

Find the following limits as $x \to 0$ or state that the limit does not exist.

a) (2 points) $\frac{\tan(x)}{x}$

b) (2 points) $\frac{x}{\cos(x)-x}$.

c) (2 points) $x \log(1+x)/\sin(x)$.

d) (2 points) $x \log(x)$.

e) (2 points) $x/(1-\exp(x))$.

Problem 12) Applications (10 points)

The cumulative distribution function on $[0, 1]$

$$F(x) = \frac{2}{\pi} \arcsin(\sqrt{x})$$

defines the **arc-sin** distribution.

a) Find the probability density function $f(x)$ on $[0, 1]$.

b) Verify that $\int_0^1 f(x)\, dx = 1$.
Remark. The arc sin distribution is important chaos theory and probability theory.

Problem 13) Data (10 points)

Find the best linear fit $y = mx$ through the data points $(3, 5), (1, 1), (-1, 1), (2, 2)$.