Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

5/7/2020: Final Exam

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ··· but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F $\sin(3\pi/2) = -1$

2) T F The cotangent function is monotonically decreasing on the open interval $(\pi/4, \pi/2)$.

3) T F The arccot function is monotonically increasing from 1 to 2.

4) T F If F is a CDF, then $F(0) = 0$

5) T F $\frac{d}{dx} \log(e^x) = 1$, where as always log is the natural log.

6) T F The limit of $\sqrt{|x|}/\sin(\sqrt{|x|})$ for $x \to 0$ exists and is equal to 1.

7) T F If we apply the l'Hospital rule for the limit $\lim_{h\to 0}[f(x+h) - f(x)]/h$ we get $f'(x)$

8) T F If $f'(1) = 0$ and $f''(0) = 1$ then f has a local minimum at $x = 0$.

9) T F The improper integral $\int_{-1}^{1} 1/\sqrt{1-x^2}\, dx$ is finite.

10) T F The function $f(x) = 1 + \sin(x^2) - x^3$ has a root in the interval $(-100, 100)$.

11) T F If a function f has a local minimum and a local maximum, then it must have a second minimum.

12) T F To any continuous function f, there exists a unique $F(x)$ such that $F'(x) = f(x)$.

13) T F If you listen to the sound $\log(1+x)\sin(10000x)$, then it gets louder and louder as time goes on.

14) T F The function $f(x) = e^{-x^2}$ has a local minimum at $x = 0$

15) T F The function $f(x) = (x^{25} - 1)/(x^5 - 1)$ has the limit 20 for $x \to 1$.

16) T F If the average cost $F(x)/x$ of an entity is extremal at $x = 2$, then we have a break-even point $f(2) = g(2)$.

17) T F If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ is called the variance of f.

18) T F The Midi function $f(s)$ gives the midi number $f(s)$ as a function of the frequency s.

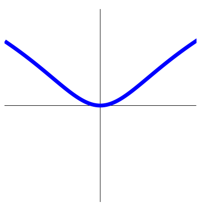
19) T F A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

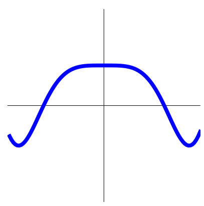
20) T F $\sin(\arcsin(1)) = 1$.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)$			
$\cos(x^2)$			
$\log(1+x^2)$			
$\exp(-x^2)$			


1)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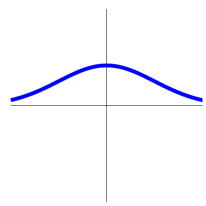
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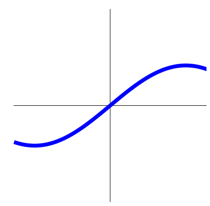
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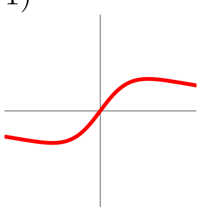
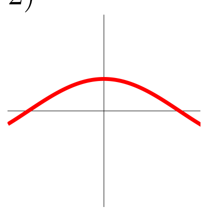
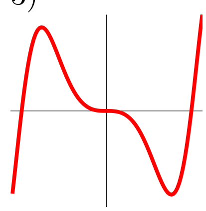
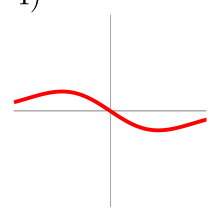
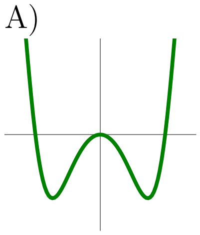
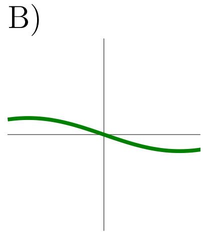

A)

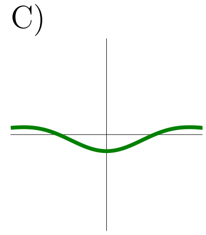

B)

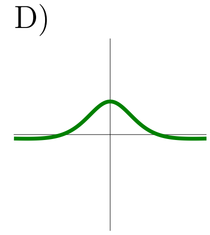

C)


D)


a)


b)


c)


d)

(5 points) Which of the following limits exists in the limit $x \to 1$. If the limit exists, enter the result

Function	Enter the limit if it exists	Check if it does not exist
$\frac{(1-x^9)}{(1-x^7)}$		
$\frac{x}{\log\|x\|}$		
$\frac{\tan(1-x)}{(1-x)}$		
$\log\|x\|/x$		
$\log(x)/\log(2x)$		
$\frac{x^2-1}{\sin(x^2-1)}$		

Problem 3) Short answer problem (10 points). No justifications are needed.

a) (3 points) Find the relation between the following functions:

function f	function g	$f = g'$	$g = f'$	none
$\log(x)$	$1/x$			
$1/x$	$-1/x^2$			
$\tan(x)$	$1/(1+x^2)$			
$\cot(x)$	$-1/\sin^2(x)$			
$\arctan(x)$	$1/\cos^2(x)$			
$\text{arccot}(x)$	$-1/(1+x^2)$			

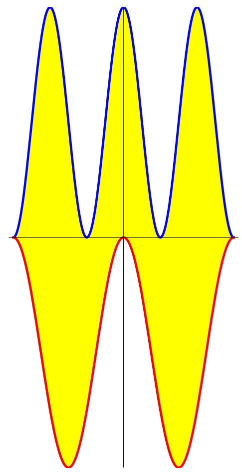
b) (3 points) We integrate $\int_0^1 f(x)\ dx$ numerically. During the numerical integration method lecture we have pointed out that some integration methods give always exact answers for quadratic functions f. For which numerical integration methods is this the case?

Integration method	The method gives the exact value for quadratic f
Archimedes (equal spacing)	
General Riemann sum	
Trapezoid method	
Simpson Method	
Simpson 3/8 Method	

c) (2 points) Formulate the "Strawberry theorem" in economics.

d) (2 points) Which mathematical theorem is involved for the "wobbly table theorem"?

Problem 4) Area computation (10 points)



Find the area enclosed by the graphs of the functions

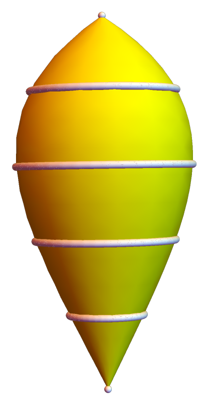
$$f(x) = \cos(2\pi x) - 1 \ .$$

and

$$g(x) = 1 + \cos(3\pi x)$$

on an interval on which f has two minima and g has three maxima. The situation is displayed in the picture.

Problem 5) Volume computation (10 points)



As kids we used to play with a **wooden top**, which is brought in motion by pulling at a rope wound around the solid. We want to the volume of such a top which at height z is a disk of radius

$$r(z) = z(1 - z^3)$$

and where the z values go from 0 to 1.

Problem 6) Improper integrals (10 points)

Which of the following improper integral are convergent? In each case either state that it is not convergent or compute the limit explicitly.

a) (2 points) $\int_1^\infty \sin(x) \, dx$

b) (2 points) $\int_1^\infty \frac{1}{x^3} \, dx$

c) (2 points) $\int_1^\infty \frac{1}{x^{1/3}} \, dx$

d) (2 points) $\int_0^1 \frac{1}{x^3} \, dx$

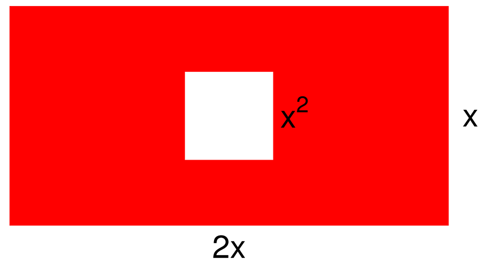
e) (2 points) $\int_0^1 \frac{1}{x^{1/3}}\, dx$

Problem 7) Extrema (10 points)

We want to find the maximal area of a rectangle of length $2x$ and height x in which a square hole of length x^2 has been taken out. The area function is

$$f(x) = 2x^2 - x^4 \ .$$

Use the second derivative test to locate the maximum.



Problem 8) Integration by parts (10 points)

a) (5 points) Compute

$$\int x^5 e^x \, dx \ .$$

b) (5 points) Evaluate the following integral. As always, $\log(x)$ is the natural log.

$$\int \log(x) x \, dx \ .$$

Problem 9) Substitution (10 points)

a) (5 points)

$$\int \frac{\sin(\log(x))}{x} \, dx \ .$$

b) (5 points)

$$\int (1-x^2)^{-5/2} dx \ .$$

Problem 10) Advanced integration (10 points)

a) (5 points) Integrate

$$\int \frac{1}{(x+1)(x+3)(x-2)(x-1)} \, dx \ .$$

b) (5 points) Use the magic **trig substitution box** to find the anti-derivative:

$$\int \frac{4dx}{\sin^3(x)} \ .$$

Here is the magic box:

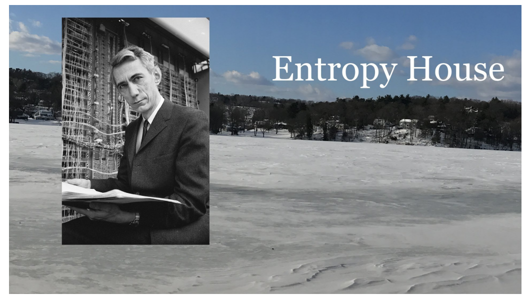
$u = \tan(x/2)$
$dx = \frac{2du}{(1+u^2)}$
$\sin(x) = \frac{2u}{1+u^2}$
$\cos(x) = \frac{1-u^2}{1+u^2}$

Problem 11) PDF's and CDF's. (10 points)

If $f(x)$ is a PDF, then

$$S = -\int_{\infty}^{\infty} f(x)\log(f(x))\,dx$$

is called the **entropy** of f.
What is the entropy of the exponential distribution, given by the function which is 0 for negative x and e^{-x} for $x \geq 0$?



The Entropy house in Winchester, MA on Mystic Lake, where Claude Shannon, the father of information theory lived. Photo: Oliver Knill, 2018.

Problem 12) Which integration method?(10 points)

a) (5 points) Find the anti-derivative of

$$\int e^{e^x} e^x \, dx \; .$$

b) (5 points) And what is the anti-derivative of

$$\int (\log(x))^2 x \, dx \; .$$

Problem 13) Applications (10 points)

a) (2 points) The CDF of the PDF. What is the CDF of the function that is given as $f(x) = 1/x^2$ if $x \geq 1$ and $f(x) = 0$ else? [Give the function for $x \geq 1$]

b) (2 points) If x is the Midi number, then $f(x) = 440e^{(x-69)/12}$ is called the [give the expression, one word.]

c) (2 points) If $f(x)$ is a PDF, then $M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$ is called a [give the expresion, one word.]

d) (2 points) For the family of functions $f_c(x) = c\cos(x)$, there is a catastrophe at $c =$ [give a number, one number.]

e) (2 points) If $(3,5)$ and $(2,7)$ are two data points, the line $y = mx$ which is the best fit minimizes the function [Give a function $f(m) = ...$]