Unit 24

Substitution

① Chain rule

$$f(x) = g(u(x))$$
$$\frac{d}{dx} f(x) = g'(u(x))\, u'(x)$$

$$f(x) = \int f'(x)\,dx = \int g'(u(x))\, u'(x)\,dx$$

Wow! We can integrate any function which has the form to the right.

- $\int \sin(x^2)\, 2x \, dx$
  $= -\cos(x^2) + C$
- $\int 2\log(x) \frac{1}{x} \, dx = \log^2(x) + C$
- $\int e^{x^2} 2x = e^{x^2} + C$

If you can spot the antiderivative, just do it!

③ $\boxed{\text{Substitution}}$

Pick a part of the formula and call it u, then compute $du = u'\,dx$ and substitute it all in

- $\int \sin(x^2)\, 2x\, dx =$

  $u = x^2$

  $du = 2x\,dx$

  $= \int \sin u \, du$

  $= -\cos u + C = -\cos(x^2) + C$

- $\int \frac{\log x}{x}\, dx = \int u\, du$

  $\boxed{\begin{array}{l} u = \log x \\ du = \frac{1}{x}\, dx \end{array}}$

  $= \frac{u^2}{2} + C$

  $= \frac{\log^2(x)}{2} + C$

- $\int e^{x^2}\, 2x\, dx = \int e^u\, du$

  $\boxed{\begin{array}{l} u = x^2 \\ du = 2x\, dx \end{array}}$

  $= e^u + C$

  $= e^{x^2} + C$

- $\int \frac{\log(\log x)}{\log x \;\; x}\, dx = \int \frac{\log u}{u}\, du$

  $\boxed{\begin{array}{l} u = \log x \\ du = \frac{1}{x}\, dx \end{array}}$

  $= \frac{\log^2 u}{2}$

$$= \frac{\log^2 \log x}{2}$$

③ Examples

- $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\log(1+x^2)+c$

$u = x^2$
$du = 2x\,dx$

- $\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{du}{1+u^2}$

$u = x^2$
$du = 2x\,dx$

$$= \frac{1}{2}\arctan(1+u^2)$$

$$= \frac{1}{2}\arctan(1+u^4)$$

- $\int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{du}{u} = -\log u + c$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\log(\cos x) + c$$

- $\int \frac{1}{\log^2 x \; x}\,dx = \int \frac{1}{u^2}\,du$

$u = \log x$
$du = \frac{1}{x}\,dx$

$$= -\frac{1}{u} + c = \frac{-1}{\log x} + c$$

- $\int \sqrt{x^4+1}\; x^3\,dx = \frac{2(x^4+1)^{3/2}}{3 \cdot 4}$

④ Tougher cases

a) $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx = \int \frac{-1}{u^3}\,du$

$u = \cos x$
$du = -\sin x\,dx$
$dx = -\dfrac{du}{\sin x}$

$\displaystyle = \frac{u^{-2}}{2} + C$

$\displaystyle = \frac{1}{2\cos^2 x} + C$

or $\displaystyle\int \frac{\tan x}{\cos^2 x}\,dx = \int u\,du = \frac{u^2}{2} + C$

$u = \tan x$
$du = \dfrac{1}{\cos^2 x}\,dx$

$\displaystyle = \frac{\tan^2 x}{2} + C$

b) $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{\sqrt{(u-1)^3}}{2\sqrt{u-1}\sqrt{u}}$

$u = x^2+1$
$du = 2x\,dx$
$dx = \dfrac{du}{2x}$

$\displaystyle = \int \frac{u-1}{2\sqrt{u}}\,du$

$\displaystyle = \int \frac{\sqrt{u}}{2} - \frac{1}{2\sqrt{u}}\,du$

$\displaystyle = \frac{u^{3/2}}{3} - \sqrt{u} + C$

$\displaystyle = \boxed{\frac{(x^2+1)^{3/2}}{3} - \sqrt{x^2+1} + C}$