Lecture 36: Review

1 Find all (possibly complex) eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 0 & 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 & 3 \\ 3 & 0 & 0 & 0 & 5 & 0 \\ 0 & 3 & 0 & 0 & 0 & 5 \end{bmatrix} .$$

Solution: The matrix A is $5 + 3Q^2$ where

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The characteristic polynomial of Q is $\lambda^5 - 1$. The roots are the eigenvalues which are$\lambda_k = e^{2\pi ik/6}$, where $k = 0, ..., 5$. The eigenvectors of Q are

$$v_k = \begin{bmatrix} 1 \\ \lambda \\ \lambda_k^2 \\ \lambda_k^3 \\ \lambda_k^4 \\ \lambda_k^5 \end{bmatrix}$$

where $k = 1, ..., 6$.

(This is always the same for these type of matrices. One can see this by writing down $Qv = \lambda v$ and set the first entry equal to 1 and get the other entries by recursion).

These eigenvectors of Q are also the eigenvectors of A. The eigenvalues of A are $5 + 3\lambda_k^2$, where $k = 1, ..., 6$.

Remarks This computation works for any n. The characteristic equation is $\lambda^n - 1$ or $1 - \lambda^n$ depending on whether n is even or odd. The roots are always the n'th roots of unity $\lambda_k = e^{2\pi ik/n}$. The eigenvectors are always the above v_k.