Lecture 29: Worksheet on Eigenvectors

1 Let

$$A = \begin{bmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{bmatrix} .$$

Find the eigenvalues and eigenvectors of A.

2 Let

$$B = A^T = \begin{bmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{bmatrix} .$$

Find the eigenvalues and eigenvectors of B.

The matrix B is called a **regular transition matrix** or simply **Markov matrix**. Find the eigenvectors of B.

3 Let's have a look at the **multiplication table** we learned as kids in first grade:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{bmatrix} .$$

Find all eigenvalues and eigenvectors of this symmetric matrix.

Solution:

The matrix A has rank 1 and so nullity 8 by the fundamental theorem of algebra. Since there is only one nonzero eigenvalue, it is equal to $\text{tr}(A) = 1 + 4 + 9 + ... + 81 = 285$ This eigenvalue has the eigenvector $[1, 2, 3, 4, 5, 6, 7, 8, 9]^T$, which also spans the image of A. The matrix is also symmetric and so a complete eigenbasis. To find the eigenspace to 0 we have to find the kernel of A which is the kernel of

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

We have

$$E_0 = \ker(A) = \ker(\text{rref}(A)) =$$

$$\text{span}\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \}$$