Lecture 35: Symmetric matrices

In this lecture, we look at the spectrum of symmetric matrices. Symmetric matrices appear in **geometry**, for example, when introducing **more general dot products** $v \cdot Av$ or in **statistics** as **correlation matrices** $\text{Cov}[X_k, X_l]$ or in quantum mechanics as **observables** or in **neural networks** as **learning maps** $x \mapsto \text{sign}(Wx)$ or in **graph theory** as **adjacency matrices**. Symmetric matrices play the same role as the **real numbers** do among the complex numbers. Their eigenvalues often have physical or geometrical interpretations. One can also calculate with symmetric matrices like with numbers: for example, we can solve $B^2 = A$ for B if A is symmetric matrix and B is square root of A.) This is not possible in general. There is no matrix B for example such that $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Recall the following definition:

A real matrix is called **symmetric** if $A^T = A$. Symmetric matrices are also called selfadjoint. For complex matrices we would ask $A^* = \overline{A}^T = A$.

1 The matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

is symmetric.

A symmetric matrix has real eigenvalues.

Proof. Extend the dot product to complex vectors by $(v, w) = \sum_i \overline{v}_i w_i$, where $\overline{v}$ is the complex conjugate. For real vectors it is the usual dot product $(v, w) = v \cdot w$. The new product has the property $(Av, w) = (v, A^T w)$ for real matrices A and $(\lambda v, w) = \overline{\lambda}(v, w)$ as well as $(v, \lambda w) = \lambda(v, w)$. Now $\overline{\lambda}(v, v) = (\lambda v, v) = (Av, v) = (v, A^T v) = (v, Av) = (v, \lambda v) = \lambda(v, v)$ shows that $\overline{\lambda} = \lambda$ because $(v, v) \neq 0$ for $v \neq 0$.

There is an orthogonal eigenbasis for a symmetric matrix A if all the eigenvalues of A all different.

Proof. Assume $Av = \lambda v$ and $Aw = \mu w$. The relation

$$\lambda(v, w) = (\lambda v, w) = (Av, w) = (v, A^T w) = (v, Aw) = (v, \mu w) = \mu(v, w)$$

is only possible if $(v, w) = 0$ if $\lambda \neq \mu$.

Spectral theorem A symmetric matrix can be diagonalized with an orthonormal matrix S.

The result is called spectral theorem. I present now an intuitive proof, which gives more insight why the result is true. The linear algebra book of Bretscher has an inductive proof.

Proof. We have seen already that if all eigenvalues are different, there is an eigenbasis and diagonalization is possible. The eigenvectors are all orthogonal and $B = S^{-1}AS$ is diagonal containing the eigenvalues. In general, we can change the matrix A to $A = A + (C - A)t$ where C is a matrix with pairwise different eigenvalues. Then the eigenvalues are different for all except finitely many t. The orthogonal matrices S_t converges for $t \to 0$ to an orthogonal matrix S and S diagonalizes A. [1]

Why could we not perturb a general matrix A_t to have disjoint eigenvalues and A_t could be diagonalized: $S_t^{-1} A_t S_t = B_t$? The problem is that S_t might become singular for $t \to 0$.

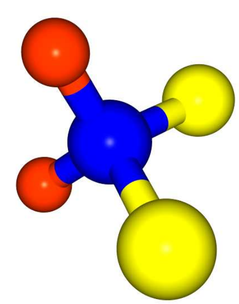
2 The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1+t \end{bmatrix}$ has the eigenvalues $1, 1+t$ and is diagalizable for $t > 0$ but not diagonalizable for $t = 0$. What happens with the diagonalization in the limit? **Solution:** Because the matrix is upper triangular, the eigenvalues are $1, 1+t$. The eigenvector to the eigenvalue 1 is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The eigenvector to the eigenvalue $1+t$ is $\begin{bmatrix} 1 \\ t \end{bmatrix}$. We see that in the limit $t \to 0$, the second eigenvector collides with the first one. For symmetric matrices, where the eigenvectors are always perpendicular to each other, such a collision can not happen.

3 The **Freon molecule** (Dichlorodifluoromethane or shortly CFC-12) CCl_2F_2 has 5 atoms. It is a CFC was used in refrigerators, solvents and propellants but contributes to ozone depletion in the atmosphere. The adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Verify that the matrix has the characteristic polynomial $x^5 - 4x^3$.

b) Find the eigenvalues of A.
c) Find the eigenvectors of A.

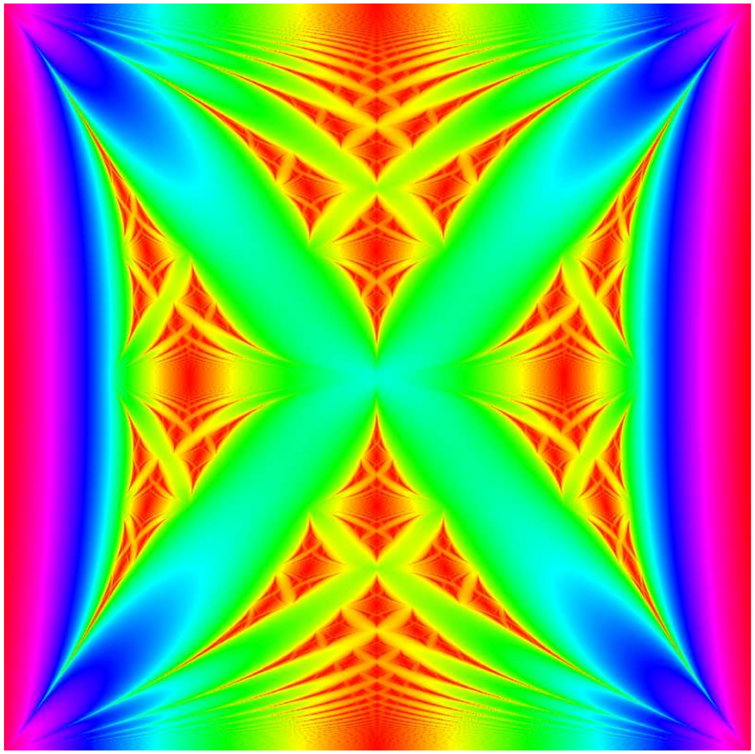


[1] This is justified by a result of Neumann-Wigner who proved that the set of symmetric matrices with simple eigenvalues is path connected and dense in the linear space of all symmetric $n \times n$ matrices.

In solid state physics or quantum mechanics, one is interested in matrices like

$$L = \begin{bmatrix} \lambda\cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda\cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda\cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda\cos(n\alpha) \end{bmatrix}$$

It appears in models describing an electron in a periodic crystal. The eigenvalues form what one calls the**spectrum** of the matrix. A physicist is interested in it because it determines what conductivity properties the system has. This depends on α.



The picture shows the eigenvalues of L for $\lambda = 2$ for $\lambda = 2$ with large n. The vertical axes is α which runs from $\alpha = 0$ at the bottom to $\alpha = 2\pi$ on the top. Due to its nature, the picture is called "Hofstadter butterfly". It has been popularized in the book "Gödel, Escher Bach" by Douglas Hofstadter.

Homework due April 27, 2011

1 For the following question, give a reason why it is true or give a counter example.
a) Is the sum of two symmetric matrices symmetric?
b) Is the product of a symmetric matrix symmetric?
c) Is the inverse of an invertible symmetric matrix symmetric?
d) If B is an arbitrary $n \times m$ matrix, is $A = B^T B$ symmetric? e) If A is similar to B and A is symmetric, then B is symmetric.
f) If A is similar to B with an orthogonal coordinate change S and A is symmetric, then B is symmetric.

2 Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 10001 & 3 & 5 & 7 & 9 & 11 \\ 1 & 10003 & 5 & 7 & 9 & 11 \\ 1 & 3 & 10005 & 7 & 9 & 11 \\ 1 & 3 & 5 & 10007 & 9 & 11 \\ 1 & 3 & 5 & 7 & 10009 & 11 \\ 1 & 3 & 5 & 7 & 9 & 10011 \end{bmatrix}.$$

As usual, document all your reasoning.

3 Which of the following classes of linear transformations are described by symmetric?
a) Reflections in the plane.
b) Rotations in the plane.
c) Orthogonal projections.
d) Shears.