Lecture 28: Eigenvalues

We have seen that $\det(A) \neq 0$ if and only if A is invertible.

The polynomial $f_A(\lambda) = \det(A - \lambda I_n)$ is called the **characteristic polynomial** of A.

The eigenvalues of A are the roots of the characteristic polynomial.

Proof. If $Av = \lambda v$,then v is in the kernel of $A - \lambda I_n$. Consequently, $A - \lambda I_n$ is not invertible and

$$\det(A - \lambda I_n) = 0 .$$

1 For the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$, the characteristic polynomial is

$$\det(A - \lambda I_2) = \det(\begin{bmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{bmatrix}) = \lambda^2 - \lambda - 6 .$$

This polynomial has the roots $3, -2$.

Let $\mathrm{tr}(A)$ denote the **trace** of a matrix, the sum of the diagonal elements of A.

For the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the characteristic polynomial is

$$\lambda^2 - \mathrm{tr}(A) + \det(A) .$$

We can see this directly by writing out the determinant of the matrix $A - \lambda I_2$. The trace is important because it always appears in the characteristic polynomial, also if the matrix is larger:

For any $n \times n$ matrix, the characteristic polynomial is of the form

$$f_A(\lambda) = (-\lambda)^n + \mathrm{tr}(A)(-\lambda)^{n-1} + \cdots + \det(A) .$$

Proof. The pattern, where all the entries are in the diagonal leads to a term $(A_{11} - \lambda) \cdot (A_{22} - \lambda)...(A_{nn} - \lambda)$ which is $(-\lambda^n) + (A_{11} + ... + A_{nn})(-\lambda)^{n-1} + ...$ The rest of this as well as the other patterns only give us terms which are of order λ^{n-2} or smaller.

How many eigenvalues do we have? For real eigenvalues, it depends. A rotation in the plane with an angle different from 0 or π has no real eigenvector. The eigenvalues are complex in that case:

2 For a rotation $A = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$ the characteristic polynomial is $\lambda^2 - 2\cos(\alpha) + 1$ which has the roots $\cos(\alpha) \pm i\sin(\alpha) = e^{i\alpha}$.

Allowing complex eigenvalues is really a blessing. The structure is very simple:

Fundamental theorem of algebra: For a $n \times n$ matrix A, the characteristic polynomial has exactly n roots. There are therefore exactly n eigenvalues of A if we count them with multiplicity.

Proof[1] One only has to show a polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ always has a root z_0 We can then factor out $p(z) = (z - z_0)g(z)$ where $g(z)$ is a polynomial of degree $(n-1)$ and use induction in n. Assume now that in contrary the polynomial p has no root. Cauchy's integral theorem then tells

$$\int_{|z|=r|} \frac{dz}{zp(z)} = \frac{2\pi i}{p(0)} \neq 0 . \quad (1)$$

On the other hand, for all r,

$$|\int_{|z|=r|} \frac{dz}{zp(z)}| \leq 2\pi r \mathrm{max}_{|z|=\mathrm{r}} \frac{1}{|zp(z)|} = \frac{2\pi}{\min_{|z|=r} p(z)} . \quad (2)$$

The right hand side goes to 0 for $r \to \infty$ because

$$|p(z)| \geq |z|^n |(1 - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_0|}{|z|^n})$$

which goes to infinity for $r \to \infty$. The two equations (1) and (2) form a contradiction. The assumption that p has no root was therefore not possible.

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then

$$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda) .$$

Comparing coefficients, we know now the following important fact:

The determinant of A is the product of the eigenvalues. The trace is the sum of the eigenvalues.

We can therefore often compute the eigenvalues

3 Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$$

Because each row adds up to 10, this is an eigenvalue: you can check that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We can also read off the trace 8. Because the eigenvalues add up to 8 the other eigenvalue is -2. This example seems special but it often occurs in textbooks. Try it out: what are the eigenvalues of

$$A = \begin{bmatrix} 11 & 100 \\ 12 & 101 \end{bmatrix} ?$$

[1] A. R. Schep. A Simple Complex Analysis and an Advanced Calculus Proof of the Fundamental theorem of Algebra. Mathematical Monthly, 116, p 67-68, 2009

4 Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

We can immediately compute the characteristic polynomial in this case because $A - \lambda I_5$ is still upper triangular so that the determinant is the product of the diagonal entries. We see that the eigenvalues are $1, 2, 3, 4, 5$.

> The eigenvalues of an upper or lower triangular matrix are the diagonal entries of the matrix.

5 How do we construct 2x2 matrices which have integer eigenvectors and integer eigenvalues? Just take an integer matrix for which the row vectors have the same sum. Then this sum is an eigenvalue to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The other eigenvalue can be obtained by noticing that the trace of the matrix is the sum of the eigenvalues. For example, the matrix $\begin{bmatrix} 6 & 7 \\ 2 & 11 \end{bmatrix}$ has the eigenvalue 13 and because the sum of the eigenvalues is 18 a second eigenvalue 5.

> A matrix with nonnegative entries for which the sum of the columns entries add up to 1 is called a **Markov matrix**.

> Markov Matrices have an eigenvalue 1.

Proof. The eigenvalues of A and A^T are the same because they have the same characteristic polynomial. The matrix A^T has an eigenvector $[1, 1, 1, 1, 1]^T$.

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$$A = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/4 & 1/3 & 1/3 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

This vector $\vec{v}$ defines an equilibrium point of the Markov process.

7 If $A = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. Then $[3/7, 4/7]$ is the equilibrium eigenvector to the eigenvalue 1.

Homework due April 13, 2011

1 a) Find the characteristic polynomial and the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix}.$$

b) Find the eigenvalues of $A = \begin{bmatrix} 100 & 1 & 1 & 1 & 1 \\ 1 & 100 & 1 & 1 & 1 \\ 1 & 1 & 100 & 1 & 1 \\ 1 & 1 & 1 & 100 & 1 \\ 1 & 1 & 1 & 1 & 100 \end{bmatrix}$.

2 a) Verify that $n \times n$ matrix has a at least one real eigenvalue if n is odd.
b) Find a 4×4 matrix, for which there is no real eigenvalue.
c) Verify that a symmetric 2×2 matrix has only real eigenvalues.

3 a) Verify that for a partitioned matrix

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

the union of the eigenvalues of A and B are the eigenvalues of C.
b) Assume we have an eigenvalue $\vec{v}$ of A use this to find an eigenvector of C. Similarly, if $\vec{w}$ is an eigenvector of B, build an eigenvector of C.
(*) Optional: Make some experiments with random matrices: The following Mathematica code computes Eigenvalues of random matrices. You will observe Girko's circular law.

```
M=1000;
A=Table[Random[]-1/2,{M},{M}];
e=Eigenvalues[A];
d=Table[Min[Table[If[i==j,10,Abs[e[[i]]-e[[j]]]],{j,M}]],{i,M}];
a=Max[d]; b=Min[d];
Graphics[Table[{Hue[(d[[j]]-a)/(b-a)],
   Point[{Re[e[[j]]],Im[e[[j]]]}]},{j,M}]]
```