Lecture 27: Discrete dynamical systems

Eigenvectors and Eigenvalues

Markets, population evolutions or ingredients in a chemical reaction are often nonlinear. A linear description often can give a good approximation and solve the system explicitly. Eigenvectors and eigenvalues provide us with the key to do so.

> A nonzero vector v is called an **eigenvector** of a $n \times n$ matrix A if $\boxed{Av = \lambda v}$ for some number λ. The later is called an **eigenvalue** of A.

We first look at real eigenvalues but also consider complex eigenvalues.

1 A vector v is an eigenvector to the eigenvalue 0 if and only if $\vec{v}$ is in the kernel of A because $A\vec{v} = 0\vec{v}$ means that $\vec{v}$ is in the kernel.

2 A rotation A in three dimensional space has an eigenvalue 1, with eigenvector spanning the axes of rotation. This vector satisfies $A\vec{v} = \vec{v}$.

3 Every standard basis vector $\vec{w}_i$ is an eigenvector if A is a diagonal matrix. For example, $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

4 For an orthogonal projection P onto a space V, every vector in V is an eigenvector to the eigenvalue 1 and every vector perpendicular to V is an eigenvector to the eigenvalue 0.

5 For a reflection R at a space V, every vector v in V is an eigenvector with eigenvalue 1. Every vector perpendicular to v is an eigenvector to the eigenvalue -1.

Discrete dynamical systems

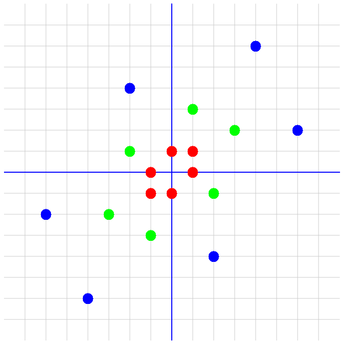
> When applying a linear map $x \mapsto Ax$ again and again, we obtain a **discrete dynamical system**. We want to understand what happens with the **orbit** $x_1 = Ax, x_2 = AAx = A^2x, x_3 = AAAx = A^3x, ...$ and find a closed formula for $A^n x$

6 The one-dimensional discrete dynamical system $x \mapsto ax$ or $x_{n+1} = ax_n$ has the solution $x_n = a^n x_0$. The value $1.03^{20} \cdot 1000 = 1806.11$ for example is the balance on a bank account which had 1000 dollars 20 years ago if the interest rate was a constant 3 percent.

7 Look at the recursion

$$u_{n+1} = u_n - u_{n-1}$$

with $u_0 = 0, u_1 = 1$. Because $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$ we have a discrete dynamical system. Lets compute some orbits: $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We see that A^6 is the identity. Every initial vector is mapped after 6 iterations back to its original starting point.



8 $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A^2\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. $A^3\vec{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. $A^4\vec{v} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ etc. Do you see a pattern?

The following example shows why eigenvalues and eigenvectors are so important:

9 If $\vec{v}$ is an eigenvector with eigenvalue λ, then $A\vec{v} = \lambda\vec{v}, A^2\vec{v} = A(A\vec{v}) = A\lambda\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$ and more generally $A^n\vec{v} = \lambda^n\vec{v}$.

> For an eigenvector, we have a closed form solution for $A^n\vec{v}$. It is $\lambda^n\vec{v}$.

10 The recursion

$$x_{n+1} = x_n + x_{n-1}$$

with $x_0 = 0$ and $x_1 = 1$ produces the **Fibonacci sequence**

$$(1, 1, 2, 3, 5, 8, 13, 21, ...)$$

This can be computed with a discrete dynamical system because

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} .$$

Can we find a formula for the n'th term?

In the third section of **Liber abbaci**, published in 1202, the mathematician Fibonacci, with real name **Leonardo di Pisa** (1170-1250) writes: **"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"**

Markov case

We will discuss the following situation a bit more in detail:

An $n \times n$ matrix is called a **Markov matrix** if all entries are nonnegative and each column adds up to 1.

11 Customers using **Apple IOS** and **Google Android** are represented by a vector $\begin{bmatrix} A \\ G \end{bmatrix}$. Each cycle 1/3 of IOS users switch to Android and 2/3 stays. Also lets assume that 1/2 of the Android OS users switch to IOS and 1/2 stay. The matrix $A = \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix}$ is a **Markov matrix**. What customer ratio do we have in the limit? The matrix A has an eigenvector $(3/5, 2/5)$ which belongs to the eigenvalue 1.

$$A\vec{v} = \vec{v}$$

means that 60 to 40 percent is the final stable distribution.

The following fact motivates to find good methods to compute eigenvalues and eigenvectors.

If $A\vec{v}_1 = \lambda_1 \vec{v}_1, A\vec{v}_2 = \lambda_2 \vec{v}_2$ and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, we have **closed form solution** $A^n\vec{v} = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2$.

Lets try this in the Fibonacci case. We will see next time how we find the eigenvalues and eigenvectors:

12 Lets try to find a number ϕ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ 1 \end{bmatrix} = \phi \begin{bmatrix} \phi \\ 1 \end{bmatrix}$$

This leads to the quadratic equation $\phi + 1 = \phi^2$ which has the solutions $\phi_+ = (1+\sqrt{5})/2$ and $\phi_- = (1-\sqrt{5})/2$. The number ϕ^+ is one of the most famous and symmetric numbers in mathematics called the **golden ratio**. We have found our eigenvalues and eigenvectors. Now find c_1, c_2 such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \phi^+ \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \phi^- \\ 1 \end{bmatrix}$$

We see $c_1 = -c_2 = 1/\sqrt{5}$. We can write

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\phi_+^n \begin{bmatrix} \phi_+ \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}\phi_-^n \begin{bmatrix} \phi_- \\ 1 \end{bmatrix}$$

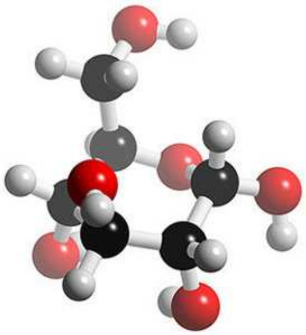
and can read off $x_n = (\phi_+^n - \phi_-^n)/\sqrt{5}$.

Homework due April 13, 2011

1 Compare Problem 52 in Chapter 7.1 of Bretscher's Book. The glucose and excess hormone concentration in your blood are modeled by a vector $\vec{v} = \begin{bmatrix} g \\ h \end{bmatrix}$. Between meals the concentration changes to $\vec{v} \to A\vec{v}$, where

$$A = \begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix} .$$

Check that $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ are eigenvectors of A. Find the eigenvalues.



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2 Compare Problem 54 in Chapter 7.1 of Bretscher's Book. The dynamical system $v_{n+1} = Av_n$ with

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

models the growth of a **lilac bush**. The vector $\vec{v} = \begin{bmatrix} n \\ a \end{bmatrix}$ models the number of new branches and the number of old branches. Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors of A. Find the eigenvalues and find the close form solution starting with $\vec{v} = [2, 3]^T$.

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3 Compare problem 50 in Chapter 7.1 of Bretscher's Book. Two interacting populations of **hares and foxes** can be modeled by the discrete dynamical system $v_{n+1} = Av_n$ with

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

Find a closed form solutions in the following three cases: a) $\vec{v}_0 = \begin{bmatrix} h_0 \\ f_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

b) $\vec{v}_0 = \begin{bmatrix} h_0 \\ f_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

c) $\vec{v}_0 = \begin{bmatrix} h_0 \\ f_0 \end{bmatrix} = \begin{bmatrix} 600 \\ 500 \end{bmatrix}$.