Lecture 26: Determinants part three

Geometry

For a $n \times n$ matrix, the determinant defines a volume of the parallelepiped spanned by the column vectors.

Since we have not defined volume in higher dimensions, we can take the absolute value of the determinant as the definition of the volume. The sign of the determinant gives additional information, it defines the **orientation**. Determinants are useful to **define** these concepts in higher dimensions. The linearity result shown last time illustrates why this makes sense: scaling one of the vector by a factor 2 for example changes the determinant by a factor 2. This also increases the volume by a factor of 2 which can be interpreted as stacking two such solids on top of each other.

1 The area of the parallelogram spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the determinant of

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

which is 4.

2 Find the volume of the parallelepiped spanned by the column vectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution: The determinant is -1. The volume is 1. The fact that the determinant is negative reflects the fact that these three vectors form a "left handed" coordinate system.

The volume of a k dimensional parallelepiped defined by the vectors $v_1, \dots, v_k$ is $\sqrt{\det(A^T A)}$.

We can take also this as the definition of the volume. Note that $A^T A$ is a square matrix.

3 The area of the parallelogram in space spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is

$$\det(A^T A) = \det(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}) = \det(\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}) = 2 \, .$$

The area is therefore $\sqrt{2}$. If you have seen multivariable calculus, you could also have computed the area using the cross product. The area is the length of the cross product $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ of the two vectors which is $\sqrt{2}$ too.

Cramer's rule

Solution to the linear system equations $A\vec{x} = \vec{b}$ can be given explicitly using determinants:

Cramers rule: If A_i is the matrix, where the column $\vec{v}_i$ of A is replaced by $\vec{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)}$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n] = \det([v_1, \dots, (Ax), \dots, v_n] = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n] = x_i \det([v_1, \dots, v_i, \dots, v_n]) = x_i \det(A)$.

4 Solve the system $5x+3y = 8, 8x+5y = 2$ using Cramer's rule. **Solution**. This linear system with $A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. We get $x = \det \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix} = 34 y = \det \begin{bmatrix} 5 & 8 \\ 8 & 2 \end{bmatrix} = -54$.

Gabriel Cramer was born in 1704 in Geneva. He worked on geometry and analysis until his death at in 1752 during a trip to France. Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébraique", where he used the method to solve systems of equations with 5 unknowns. According to a short biography of Cramer by J.J O'Connor and E F Robertson, the rule had been used already before by other mathematicians. Solving systems with Cramer's formulas is slower than by Gaussian elimination. But it is ueful for example if the matrix A or the vector b depends on a parameter t, and we want to see how x depends on the parameter t. One can find explicit formulas for $(d/dt)x_i(t)$ for example.

Cramer's rule leads to an explicit formula for the inverse of a matrix inverse of a matrix:

Let A_{ij} be the matrix where the i'th row and the j'th column is deleted. $B_{ij} = (-1)^{i+j}\det(A_{ji})$ is called the **classical adjoint** or **adjugate** of A. The determinant of the classical adjugate is called **minor**.

$$[A^{-1}]_{ij} = (-1)^{i+j} \frac{\det(A_{ji})}{\det(A)}$$

Proof. The columns of A^{-1} are the solutions of

$$A\vec{x} = \vec{e}_j$$

where $\vec{e}_j$ are basis vectors.

Don't confuse the classical adjoint with the **transpose** A^T. The classical adjoint is the transpose of the matrix where the i'th row and j'th column is deleted. The mix up is easy to do since the transpose is often also called the **adjoint**.

5 $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \\ 6 & 0 & 7 \end{bmatrix}$ has $\det(A) = -17$ and we get $A^{-1} = \begin{bmatrix} 14 & -21 & 10 \\ -11 & 8 & -3 \\ -12 & 18 & -11 \end{bmatrix} /(-17)$:

$B_{11} = (-1)^2 \det \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} = 14$. $B_{12} = (-1)^3 \det \begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix} = -21$. $B_{13} = (-1)^4 \det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 10$.

$B_{21} = (-1)^3 \det \begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} = -11$. $B_{22} = (-1)^4 \det \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} = 8$. $B_{23} = (-1)^5 \det \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = -3$.

$B_{31} = (-1)^4 \det \begin{bmatrix} 5 & 2 \\ 6 & 0 \end{bmatrix} = -12$. $B_{32} = (-1)^5 \det \begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix} = 18$. $B_{33} = (-1)^6 \det \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} = -11$.

Random matrices

If the entries of a matrix are random variables with a continuous distribution, then the determinant is nonzero with probability one.

If the entries of a matrix are random variables which have the property that $P[X = x] = p > 0$ for some x, then there is a nonzero probability that the determinant is zero.

Proof. We have with probability p^{2n} that the first two rows have the same entry x.

What is the distribution of the determinant of a random matrix? These are questions which are hard to analyze theoretically. Here is an experiment: we take random 100×100 matrices and look at the distribution of the logarithm of the determinant.

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M=100; T:=Log[Abs[Det[Table[Random[NormalDistribution[0,1]],{M},{M}]]]];
s=Table[T,{1000}]; S=Histogram[s]
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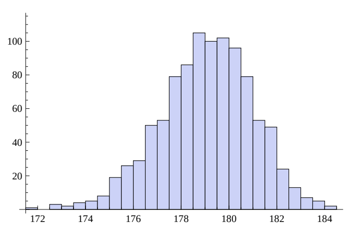


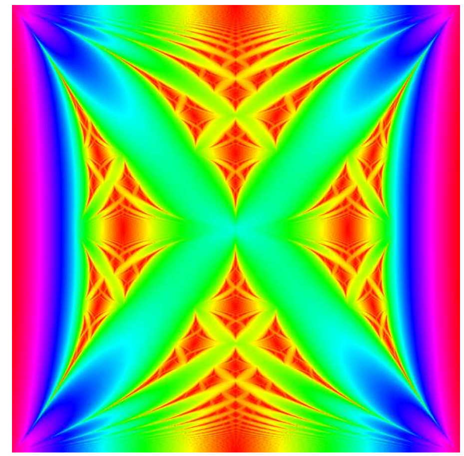
Applications of determinants

In s̊olid state physics, one is interested in the function $f(E) = \det(L - EI_n)$, where

$$L = \begin{bmatrix} \lambda\cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda\cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda\cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda\cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the **spectrum** of the quantum mechanical system. A physicist is interested in the rate of change of $f(E)$ or its dependence on λ when E is fixed. .

The graph to the left shows the function $E \mapsto \log(|\det(L - EI_n)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on α. It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher Bach" by Douglas Hofstadter.



Homework due April 6, 2011

1 Find the following determinants

a) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$ b) $A = \begin{bmatrix} 2 & 1 & 4 & 4 & 2 \\ 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

2 Find the following determinants

a) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

3 Find a 4×4 matrix A with entries $0, +1$ and -1 for which the determinant is maximal. Hint. Think about the volume. How do we get a maximal volume?