Lecture 24: Determinants

In this lecture, we define the determinant for a general $n \times n$ matrix and look at the Laplace expansion method to compute them. A determinant attaches a number to a square matrix, which determines a lot about the matrix, like whether the matrix is invertible.

The 2×2 case

The determinant of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as $\det(A) = ad - bc$.

We have seen that this is useful for the inverse:

$$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc} .$$

This formula shows:

A 2×2 matrix A is invertible if and only if $\det(A) \neq 0$. The determinant determines the invertibility of A.

1 $\det(\begin{bmatrix} 5 & 4 \\ 2 & 1 \end{bmatrix}) = 5 \cdot 1 - 4 \cdot 2 = -3$.

We also see already that the determinant changes sign if we flip rows, that the determinant is linear in each of the rows.

We can write the formula as a sum over all permutations of $1, 2$. The first permutation $\pi = (1, 2)$ gives the sum $A_{1,\pi(1)} A_{2,pi(2)} = A_{1,1} A_{2,2} = ad$ and the second permutation $\pi = (2, 1)$ gives the sum $A_{1,\pi(1)} A_{2,\pi(2)} = A_{1,2} - A_{2,1} = bc$. The second permutation has $|\pi|$ upcrossing and the sign $(-1)^{|\pi|} = -1$. We can write the above formula as $\sum_\pi (-1)^{|\pi|} A_{1\pi(1)} A_{2\pi(2)}$.

$$\begin{bmatrix} \mathrm{a} & b \\ c & \mathrm{d} \end{bmatrix} - \begin{bmatrix} a & \mathrm{b} \\ \mathrm{c} & d \end{bmatrix}$$

The 3×3 case

The determinant of a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is defined as $aei + bfg + cdh - ceg - bdi - afh$.

We can write this as a sum over all permutations of $\{1, 2, 3\}$. Each permutation produces a "pattern" along we multiply the matrix entries. The patterns π with an even number $|\pi|$ of upcrossings are taken with a positive sign the other with a negative sign.

$$\begin{bmatrix} \mathrm{a} & b & c \\ d & \mathrm{e} & f \\ g & h & \mathrm{i} \end{bmatrix} + \begin{bmatrix} a & \mathrm{b} & c \\ d & e & \mathrm{f} \\ \mathrm{g} & h & i \end{bmatrix} + \begin{bmatrix} a & b & \mathrm{c} \\ \mathrm{d} & e & f \\ g & \mathrm{h} & i \end{bmatrix} - \begin{bmatrix} a & b & \mathrm{c} \\ d & \mathrm{e} & f \\ \mathrm{g} & h & i \end{bmatrix} - \begin{bmatrix} a & \mathrm{b} & c \\ \mathrm{d} & e & f \\ g & h & \mathrm{i} \end{bmatrix} - \begin{bmatrix} \mathrm{a} & b & c \\ d & e & \mathrm{f} \\ \mathrm{g} & h & i \end{bmatrix}$$

2 $\det(\begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}) = 2 - 4 = -2$.

The general definition

A **permutation** is an invertible transformation $\{1, 2, 3, \dots, n\}$ onto itself. We can visualize it using the **permutation matrix** which is everywhere 0 except $A_{i,\pi(i)} = 1$.

There are $n! = n(n-1)(n-2) \cdots 2 \cdot 1$ permutations.

3 For $\pi = (6, 4, 2, 5, 3, 1)$ if $\pi(1) = 6, \pi(2) = 4, \pi(3) = 2, \pi(4) = 5, \pi(5) = 3, \pi(6) = 1$ we have the permutation matrix

$$P_\pi = \begin{bmatrix} & & & & & 1 \\ & & & 1 & & \\ & 1 & & & & \\ & & & & 1 & \\ & & 1 & & & \\ 1 & & & & & \end{bmatrix} .$$

It has $|\pi| = 5 + 2 + 3 + 1 + 1 = 12$ up-crossings. The determinant of a matrix which has everywhere zeros except $A_{i\pi(j)} = 1$ is the number $(-1)^{|\pi}$ which is called the **sign** of the permutation.

The **determinant** of a $n \times n$ matrix A is defined as the sum

$$\sum_\pi (-1)^{|\pi|} A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)} ,$$

where π is a permutation of $\{1, 2, \dots, n\}$ and $|\pi|$ is the number of up-crossings.

4

$$\det(A) = \det(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 11 & 0 & 0 & 0 \\ 13 & 0 & 0 & 0 & 0 & \end{bmatrix}) = 2*3*5*7*11*13\,.$$

The determinant of an upper triangular matrix or lower triangular matrix is the product of the diagonal entries.

Laplace expansion

This Laplace expansion is a convenient way to sum over all permutations. We group the permutations by taking first all the ones where the first entry is 1, then the one where the first entry is 2 etc. In that case we have a permutation of $(n-1)$ elements. the sum over these entries produces a determinant of a smaller matrix.

For each entry a_{j1} in the first column form the $(n-1)\times(n-1)$ matrix B_{j1} which does not contain the first and j'th row. The determinant of B_{j1} is called a **minor**.

Laplace expansion $\det(A) = (-1)^{1+1}A_{11}\det(B_{i1}) + \cdots + (-1)^{1+n}A_{n1}\det(B_{n1})$

5 Find the determinant of

$$\begin{bmatrix} 0 & 0 & 7 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have two nonzero entries in the first column.

$$\begin{aligned} \det(A) &= (-1)^{2+1}8\det\begin{bmatrix} 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix} + (-1)^{4+1}3\det\begin{bmatrix} 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= -8(2*7*1*5*1) + -3(2*7*0*5*1) = -560 \end{aligned}$$

6 Find the determinant of

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The answer is -1.

7 Find the determinant of

$$A = \begin{bmatrix} 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Answer: 60.

Homework due April 6, 2011

1 Find the determinant of the following matrix

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 4 & 7 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \end{bmatrix}.$$

2 Give the reason in terms of permutations why the determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$.

Example $\det(\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}) = 2\cdot 12 = 24$.

3 Find the determinant of the diamond matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 0 & 8 & 8 & 8 & 8 & 8 & 0 & 0 \\ 0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 \\ 0 & 0 & 8 & 8 & 8 & 8 & 8 & 0 & 0 \\ 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hint. Do not compute too much. Investigate what happens with the determinant if you switch two rows.