Lecture 18: Projections

A linear transformation P is called an **orthogonal projection** if the image of P is V and the kernel is perpendicular to V and $P^2 = P$.

Orthogonal projections are useful for many reasons. First of all however:

In an orthonormal basis $P = P^T$. The point Px is the point on V which is closest to x.

Proof. $Px - x$ is perpendicular to Px because

$$(Px - x) \cdot Px = Px \cdot Px - x \cdot Px = P^2 x \cdot x - x \cdot Px = Px \cdot x - x \cdot Px = 0 .$$

We have used that $P^2 = P$ and $Av \cdot w = v \cdot A^T w$.

For an orthogonal projection P there is a basis in which the matrix is diagonal and contains only 0 and 1.

Proof. Chose a basis $\mathcal{B}_\infty$ of the kernel of P and a basis $\mathcal{B}_\in$ of V, the image of P. Since for every $\vec{v} \in \mathcal{B}_1$, we have $Pv = 0$ and for every $\vec{v} \in \mathcal{B}_2$, we have $Pv = v$, the matrix of P in the basis $\mathcal{B}_1 \cup \mathcal{B}_2$ is diagonal.

1 The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} /3$$

is a projection onto the one dimensional space spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

2 The matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a projection onto the xy-plane.

3 If V is a line containing the unit vector $\vec{v}$ then $Px = v(v \cdot x)$, where $\cdot$ is the dot product. Writing this as a matrix product shows $Px = AA^T x$ where A is the $n \times 1$ matrix which contains $\vec{v}$ as the column. If v is not a unit vector, we know from multivariable calculus that $Px = v(v \cdot x)/|v|^2$. Since $|v|^2 = A^T A$ we have $Px = A(A^T A)^{-1} A^T x$.

How do we construct the matrix of an orthogonal projection? Lets look at an other example

4 Let v, w be two vectors in three dimensional space which both have length 1 and are perpendicular to each other. Now

$$Px = (v \cdot x)\vec{v} + (w \cdot x)\vec{w} .$$

We can write this as AA^T, where A is the matrix which contains the two vectors as column vectors. For example, if $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} /\sqrt{3}$ and $w = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} /\sqrt{6}$, then

$$P = AA^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

For any matrix, we have $(\mathrm{im}(A))^\perp = \ker(A^T)$.

Remember that a vector is in the kernel of A^T if and only if it is orthogonal to the rows of A^T and so to the columns of A. The kernel of A^T is therefore the orthogonal complement of $\mathrm{im}(A)$ for any matrix A:

For any matrix, we have $\ker(A) = \ker(A^T A)$.

Proof. $\subset$ is clear. On the other hand $A^T A v = 0$ means that Av is in the kernel of A^T. But since the image of A is orthogonal to the kernel of A^T, we have $A\vec{v} = 0$, which means $\vec{v}$ is in the kernel of A.

If V is the image of a matrix A with trivial kernel, then the projection P onto V is

$$Px = A(A^T A)^{-1} A^T x .$$

Proof. Let y be the vector on V which is closest to Ax. Since $y - Ax$ is perpendicular to the image of A, it must be in the kernel of A^T. This means $A^T(y - Ax) = 0$. Now solve for x to get the **least square solution**

$$x = (A^T A)^{-1} A^T y .$$

The projection is $Ax = A(A^T A)^{-1} A^T y$.

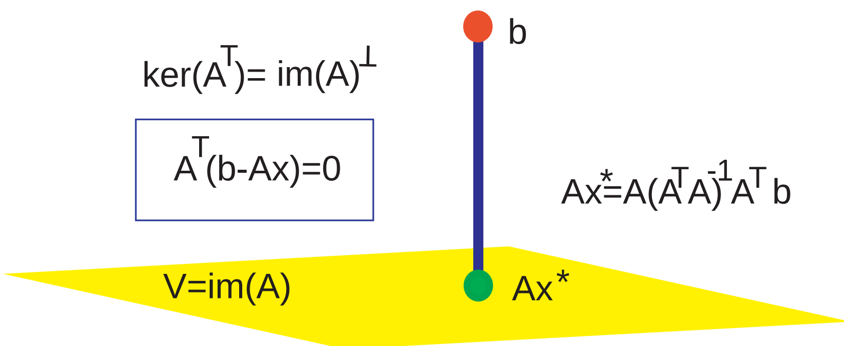
5 Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$. The orthogonal projection onto $V = \mathrm{im}(A)$ is $\vec{b} \mapsto A(A^T A)^{-1} A^T \vec{b}$. We have $A^T A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ and $A(A^T A)^{-1} A^T = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

For example, the projection of $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is $\vec{x}^* = \begin{bmatrix} 2/5 \\ 4/5 \\ 0 \end{bmatrix}$ and the distance to $\vec{b}$ is $1/\sqrt{5}$. The point $\vec{x}^*$ is the point on V which is closest to $\vec{b}$.

6 Let $A = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. Problem: find the matrix of the orthogonal projection onto the image of A.

The image of A is a one-dimensional line spanned by the vector $\vec{v} = (1, 2, 0, 1)$. We calculate $A^T A = 6$. Then

$$A(A^T A)^{-1} A^T = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} /6 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} /6 \,.$$



Homework due March 23, 2011

1 a) Find the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ onto the subspace of R^4 spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} .$$

b) Find the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto the subspace of R^5 which has the basis

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \} .$$

2 Let A be a matrix with trivial kernel. Define the matrix $P = A(A^T A)^{-1} A^T$.
a) Verify that we have $P^T = P$.
b) Verify that we have $P^2 = P$.
For this problem, just use the basis properties of matrix algebra like $(AB)^T = B^T A^T$.

3 a) Verify that the identity matrix is a projection.
b) Verify that the zero matrix is a projection.
c) Find two orthogonal projections P, Q such that $P + Q$ is not a projection.
d) Find two orthogonal projections P, Q such that PQ is not a projection.