Lecture 16: Coordinates

A basis $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\}$ of R^n defines the matrix $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. It is called the **coordinate transformation matrix** of the basis.

By definition, the matrix S is invertible: the linear independence of the column vectors implies S has no kernel. By the rank-nullety theorem, the image is the entire space R^n.

If $\vec{x}$ is a vector in R^n and $\vec{x} = c_1\vec{v}_1 + \cdots + c_n v_n$, then c_i are called the **$\mathcal{B}$-coordinates** of $\vec{v}$.

We have seen that such a representation is unique if the basis is fixed.

We write $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$, we have $\vec{x} = S([\vec{x}]_{\mathcal{B}})$.

The $\mathcal{B}$-coordinates of $\vec{x}$ are obtained by applying S^{-1} to the coordinates of the standard basis:

$[\vec{x}]_{\mathcal{B}} = S^{-1}(\vec{x})$

This just rephrases that $S[\vec{x}]_{\mathcal{B}} = \vec{x}$. Remember the column picture. The left hand side is just $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ where the v_j are the column vectors of S.

1 If $\vec{x} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, then $4, -2, 3$ are the standard coordinates. With the standard basis $\mathcal{B} = \{e_1, e_2, e_3\}$ we have $\vec{x} = 4e_1 - 2e_2 + 3e_3$.

2 If $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then $S = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. A vector $\vec{v} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ has the coordinates

$$S^{-1}\vec{v} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

Indeed, as we can check, $-3\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

3 Find the coordinates of $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.
We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Therefore $[v]_{\mathcal{B}} = S^{-1}\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Indeed $-1\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis in $\mathbf{R}^n$ and T is a linear transformation on $\mathbf{R}^n$, then the $\mathcal{B}$-matrix of T is

$$B = \begin{bmatrix} | & \dots & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & \dots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & \dots & | \end{bmatrix}.$$

4 Find a clever basis for the reflection of a light ray at the line $x + 2y = 0$. **Solution:** Use one vector in the line and an other one perpendicular to it: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. We achieved so $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = S^{-1}AA$ with $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

If A is the matrix of a transformation in the standard coordinates then

$$B = S^{-1}AS$$

is the matrix in the new coordinates.

The transformation S^{-1} maps the coordinates from the standard basis into the coordinates of the new basis. In order to see what a transformation A does in the new coordinates, we first map it back to the old coordinates, apply A and then map it back again to the new coordinates.

$$\begin{array}{ccc} \vec{v} & \stackrel{S}{\leftarrow} & \vec{w} = [\vec{v}]_{\mathcal{B}} \\ A \downarrow & & \downarrow B \\ A\vec{v} & \stackrel{S^{-1}}{\rightarrow} & B\vec{w} \end{array}$$

5 Let T be the reflection at the plane $x + 2y + 3z = 0$. Find the transformation matrix B in the basis $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. Because $T(\vec{v}_1) = \vec{v}_1 = [\vec{e}_1]_{\mathcal{B}}, T(\vec{v}_2) = \vec{v}_2 = [\vec{e}_2]_{\mathcal{B}}, T(\vec{v}_3) = -\vec{v}_3 = -[\vec{e}_3]_{\mathcal{B}}$, the solution is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Two matrices A, B which are related by $B = S^{-1}AS$ are called **similar**.

6 If A is similar to B, then $A^2 + A + 1$ is similar to $B^2 + B + \mathbf{1}$. $B = S^{-1}AS, B^2 = S^{-1}B^2S, S^{-1}S = \mathbf{1}$, $S^{-1}(A^2 + A + 1)S = B^2 + B + \mathbf{1}$.

7 If A is a general 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, What is $S^{-1}AS$? **Solution:** $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$. Both the rows and columns have switched. This example shows that the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ are similar.

Homework due March 9, 2011

1 Find the $\mathcal{B}$-matrix B of the linear transformation which is given in standard coordinates as

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

if $\mathcal{B} = \{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$.

2 Let V be the plane spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Find the matrix A of reflection at the plane V by using a suitable coordinate system.

3 a) Find a basis which describes best the points in the following lattice: We aim to describe the lattice points with integer coordinates (k, l).
b) Once you find the basis, draw all the points which have (x, y) coordinates in the disc $x^2 + y^2 \leq 10$