Lecture 15: Dimension

Remember that $X \subset R^n$ is called a **linear space** if $\vec{0} \in X$ and if X is closed under addition and scalar multiplication. Examples are $\mathbf{R}^n, X = \ker(A), X = \mathrm{im}(A)$, or the row space of a matrix. In order to describe linear spaces, we had the notion of a basis:
$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \subset X$ is a basis if two conditions are satisfied: $\mathcal{B}$ is **linear independent** meaning that $c_1\vec{v}_1 + ... + c_n\vec{v}_n = 0$ implies $c_1 = \dots = c_n = 0$. Then $\mathcal{B}$ **span** X: $\vec{v} \in X$ then $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$. The spanning condition for a basis assures that there are **enough** vectors to represent any other vector, the linear independence condition assures that there are **not too many** vectors. Every $\vec{v} \in X$ can be written uniquely as a sum $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ of basis vectors.

The **dimension** of a linear space V is the number of basis vectors in V.

The dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. For example: a line in the plane and a line embedded in space have both the dimension 1.

1 The dimension of R^n is n. The standard basis is

$$\begin{bmatrix} 1 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ ... \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ ... \\ 1 \end{bmatrix}.$$

2 The dimension of $\{0\}$ is zero. The dimension of any line 1. The dimension of a plane is 2.

3 The dimension of the image of a matrix is the number of pivot columns. We can construct a basis of the kernel and image of a linear transformation $T(x) = Ax$ by forming $B = \mathrm{rref}A$. The set of Pivot columns in A form a basis of the image of T.

4 The dimension of the kernel of a matrix is the number of free variables. It is also called **nullety**. A basis for the kernel is obtained by solving $Bx = 0$ and introducing free variables for the redundant columns.

Given a basis $\mathcal{A} = \{v_1, ..., v_n\}$ and a basis $\mathcal{B} = \{w_1, ..., .w_m\}$ of X, then $m = n$.

Lemma: if q vectors $\vec{w}_1, ..., \vec{w}_q$ span X and $\vec{v}_1, ..., \vec{v}_p$ are linearly independent in X, then $q \geq p$.

Assume $q < p$. Because $\vec{w}_i$ span, each vector $\vec{v}_i$ can be written as $\sum_{j=1}^q a_{ij}\vec{w}_j = \vec{v}_i$. Now do Gauss-Jordan elimination of the augmented $(p \times (q+n))$-matrix to this system: $\begin{vmatrix} a_{11} & \dots & a_{1q} & | & \vec{v}_1^T \\ \dots & \dots & \dots & | & \dots \\ a_{p1} & \dots & a_{pq} & | & \vec{v}_p^T \end{vmatrix}$, where $\vec{v}_i^T$ is the vector $\vec{v}_i$ written as a row vector. Each row of A of this $\begin{bmatrix} A|b \end{bmatrix}$ contains some nonzero entry. We end up with a matrix, which contains a last row $\begin{vmatrix} 0 & ... & 0 & | & b_1\vec{w}_1^T + ... + b_q\vec{w}_q^T \end{vmatrix}$ showing that $b_1\vec{w}_1^T + \dots + b_q\vec{w}_q^T = 0$. Not all b_j are zero because we had to eliminate some nonzero entries in the last row of A. This nontrivial relation of $\vec{w}_i^T$ (and the same relation for column vectors $\vec{w}$) is a contradiction to the linear independence of the $\vec{w}_j$. The assumption $q < p$ can not be true.

To prove the proposition, use the lemma in two ways. Because $\mathcal{A}$ spans and $\mathcal{B}$ is linearly independent, we have $m \leq n$. Because $\mathcal{B}$ spans and $\mathcal{A}$ is linearly independent, we have $n \leq m$.

The following theorem is also called the **rank-nullety** theorem because $\dim(\mathrm{im}(A))$ is the rank and $\dim(\ker(A))\dim(\ker(A))$ is the nullety.

Fundamental theorem of linear algebra: Let $A : \mathbf{R}^m \to \mathbf{R}^n$ be a linear map.

$$\boxed{\dim(\ker(A)) + \dim(\mathrm{im}(A)) = m}$$

There are n columns. $\dim(\ker(A))$ is the number of columns without leading 1, $\dim(\mathrm{im}(A))$ is the number of columns with leading 1.

5 If A is an invertible $n \times n$ matrix, then the dimension of the image is n and that the $\dim(\ker)(A) = 0$.

6 The first grade **multiplication table** matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{bmatrix}.$$

has rank 1. The nullety is therefore 8.

7 Are there a 4×4 matrices A, B of ranks 3 and 1 such that $\mathrm{ran}(AB) = 0$? **Solution**. Yes, we can even find examples which are diagonal.

8 Is there 4×4 matrices A, B of rank 3 and 1 such that $\mathrm{ran}(AB) = 2$? **Solution**. No, the kernel of B is three dimensional by the rank-nullety theorem. But this also means the kernel of AB is three dimensional (the same vectors are annihilated). But this implies that the rank of AB can maximally be 1.

The rank of AB can not be larger than the rank of A or the rank of B.
The nullety of AB can not be smaller than the nullety of B.

We end this lecture with an informal remark about fractal dimension:
Mathematicians study objects with non-integer dimension since the early 20'th century. The topic became fashion in the 1980'ies, when mathematicians started to generate fractals on computers. To define fractals, the notion of dimension is extended. Here is an informal definition which can be remembered easily and allows to compute the dimension of most "star fractals" you find on

the internet when searching for fractals. It assumes that X is a bounded set. You can pick up this definition also in the Startreck movie (2009) when little Spock gets some math and ethics lectures in school. It is the simplest definition and also called box counting dimension in the math literature on earth.

Assume we can cover X with $n = n(r)$ cubes of size r and not less. The **fractal dimension** is defined as the limit

$$\dim(X) = \frac{-\log(n)}{\log(r)}$$

as $r \to 0$.

For linear spaces X, the fractal dimension of X intersected with the unit cube agrees with the usual dimension in linear algebra.

Proof. Take a basis $\mathcal{B} = \{v_1, \dots, v_m\}$ in X. We can assume that this basis vectors are all orthogonal and each vector has length 1. For given $r > 0$, place cubes at the lattice points $\sum_{j=1}^m k_j r v_j$ with integer k_j. This covers the intersection X with the unit cube with (C/r^m) cubes where $1/\sqrt{m} \leq C \leq \sqrt{m}$. The dimension of X is

$$\dim(X) = \log(C/r^m)/\log(r) = \log(C)/\log(r) + m$$

which converges to m as $r \to 0$.

9 We cover the **unit interval** $[0, 1]$ with $n = 1/r$ intervals of length r. Now,

$$\dim(X) = \frac{-\log(1/r)}{\log(r)} = 1 .$$

10 We cover the **unit square** with $n = 1/r^2$ squares of length r. Now,

$$\dim(X) = \frac{-\log(1/r^2)}{\log(r)} = 2 .$$

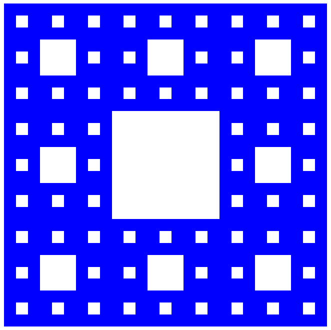
11 The **Cantor set** is obtained recursively by dividing intervals into 3 pieces and throwing away the middle one. We can cover the Cantor set with $n = 2^k$ intervals of length $r = 1/3^k$ so that

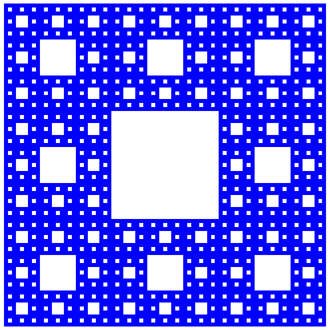
$$\dim(X) = \frac{-\log(2^k)}{\log(1/3^k)} = \log(2)/\log(3) .$$

12 The **Shirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and throwing away the middle one, repeating this procedure with each of the squares etc. At the k'th step, we need $n = 8^k$ squares of length $r = 1/3^k$ to cover X. The dimension is

$$\dim(X) = \frac{-\log(8^k)}{\log(1/3^k)} = \log(8)/\log(3) .$$

This is smaller than $2 = \log(9)/\log(3)$ but larger than $1 = \log(3)/\log(3)$.





Homework due March 9, 2011

1 a) Give an example of a 5×6 matrix with $dim(ker(A)) = 3$ or argue why it does not exist.
b) Give an example 5×8 matrix with $dim(ker(A) = 2$ or argue why it does not exist.

2 a) Find a basis for the subspace of all vectors in R^5 satisfying

$$x_1 + 2x_2 + 3x_3 - x_4 + x_5 = 0 .$$

b) Find a basis for the space spanned by the rows of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 4 & 6 & 7 \end{bmatrix} .$$

3 a) Assume two linear subspaces V, W of R^m have the property that $V \cap W = \{0\}$ and such that every vector in R^m can be written as $x + y$ with $x \in V, y \in W$. Find a formula which relates $\dim(V), \dim(W)$ and m.
b) Assume now that $V \cap W$ is 1 dimensional. What is the relation between $\dim(V)$, $\dim(W)$ and m.