Lecture 13: Image and Kernel

The image of a matrix

If $T : \mathbf{R}^m \to \mathbf{R}^n$ is a linear transformation, then $\{T(\vec{x}) \mid \vec{x} \in \mathbf{R}^m \}$ is called the **image** of T. If $T(\vec{x}) = A\vec{x}$, where A is a matrix then the image of T is also called the image of A. We write $\text{im}(A)$ or $\text{im}(T)$.

1 The map $T(x,y,z) = (x,y,0)$ maps the three dimensional space into itself. It is linear because we can find a matrix A for which $T(\vec{x}) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The image of T is the xy-plane.

2 If $T(x,y) = (\cos(\phi)x - \sin(\phi)y, \sin(\phi)x + \cos(\phi)y)$ is a rotation in the plane, then the image of T is the whole plane.

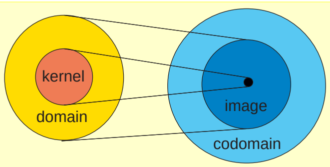
3 The averaging map $T(x,y,z) = (x+y+z)/3$ from R^3 to R has as image the entire real axes $\mathbf{R}$.

The **span** of vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbf{R}^n$ is the set of all linear combinations $c_1\vec{v}_1 + \dots c_k\vec{v}_k$.

4 The span of the standard basis vectors e_1, e_2 is the xy-plane.

A subset V of R^n is called a **linear space** if it is closed under addition scalar multiplication and contains 0.

The image of a linear transformation $\vec{x} \mapsto A\vec{x}$ is the span of the column vectors of A. The image is a linear space.



How do we compute the image? If we are given a matrix for the transformation, then the image is the span of the column vectors. But we do not need all of them in general.

A column vector of A is called a **pivot column** if it contains a leading one after row reduction. The other columns are called **redundant columns**.

The pivot columns of A span the image of A.

Proof. You can see this by deleting the other columns. The new matrix B still allows to solve $Bx = b$ if $Ax = b$ could be solved.

5 Find the image of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}.$$

The kernel of a matrix

If $T : \mathbf{R}^m \to \mathbf{R}^n$ is a linear transformation, then the set $\{x \mid T(x) = 0 \}$ is called the **kernel** of T. These are all vectors which are annihilated by the transformation. If $T(\vec{x}) = A\vec{x}$, then the kernel of T is also called the **kernel of** A. The kernel of A are all solutions to the linear system $Ax = 0$. We write $\ker(A)$ or $\ker(T)$.

6 The kernel of $T(x,y,z) = (x,y,0)$ is the z-axes. Every vector $(0,0,z)$ is mapped to 0.

7 The kernel of a rotation in the plane consists only of the zero point.

8 The kernel of the averaging map consists of all vector (x,y,z) for which $x+y+z = 0$. The kernel is a plane. In the language of random variables, the kernel of T consists of the centered random variables.

Also the kernel of a matrix A is a linear space.

How do we compute the kernel? Just solve the linear system of equations $A\vec{x} = \vec{0}$. Form $\text{rref}(A)$. For every column without leading 1 we can introduce a **free variable** s_i. If $\vec{x}$ is the solution to $A\vec{x}_i = 0$, where all s_j are zero except $s_i = 1$, then $\vec{x} = \sum_j s_j \vec{x}_j$ is a general vector in the kernel.

9 Find the kernel of $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{bmatrix}$. Gauss-Jordan elimination gives: $B = \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are two pivot columns and one redundant column. The equation $B\vec{x} = 0$ is equivalent to the system $x + 3y = 0, z = 0$. After fixing $z = 0$, can chose $y = t$ freely and obtain from the first equation $x = -3t$. Therefore, the kernel consists of vectors $t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

Homework due March 2, 2011

1 Find the image and kernel of the **chess matrix**:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} .$$

2 Find the image and kernel of the following **Pascal triangle matrix**:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{bmatrix} .$$

3 We work on the error correcting code as in the book (problem 53-54 in 3.1). Your task is to do the encoding and decoding using the initials from your name and write in one sentence using the terminology of "image" and "kernel", what the essence of this error correcting code is.

Step I) Encoding. To do so, we encode the letters of the alphabet by pairs of three vectors containing zeros and ones:

$$\begin{array}{lll}
A = (0,0,0,1),(0,0,0,1) & B = (0,0,0,1),(0,0,1,0) & C = (0,0,0,1),(0,0,1,1) \\
D = (0,0,0,1),(0,1,0,1) & E = (0,0,0,1),(0,1,1,0) & F = (0,0,0,1),(0,1,1,1) \\
G = (0,0,0,1),(1,0,0,1) & H = (0,0,0,1),(1,0,1,0) & I = (0,0,0,1),(1,0,1,1) \\
J = (0,0,0,1),(1,1,0,1) & K = (0,0,0,1),(1,1,1,0) & L = (0,0,0,1),(1,1,1,1) \\
M = (0,0,1,0),(0,0,0,1) & N = (0,0,1,0),(0,0,1,0) & O = (0,0,1,0),(0,0,1,1) \\
P = (0,0,1,0),(0,1,0,1) & Q = (0,0,1,0),(0,1,1,0) & R = (0,0,1,0),(0,1,1,1) \\
S = (0,0,1,0),(1,0,0,1) & T = (0,0,1,0),(1,0,1,0) & U = (0,0,1,0),(1,0,1,1) \\
V = (0,0,1,0),(1,1,0,1) & W = (0,0,1,0),(1,1,1,0) & X = (0,0,1,0),(1,1,1,1) \\
Y = (0,0,1,1),(1,0,0,1) & Z = (0,0,1,1),(1,0,1,0) & ? = (0,0,1,1),(1,0,1,1) \\
! = (0,0,1,1),(1,0,0,1) & . = (0,0,1,1),(1,0,1,0) & , = (0,0,1,1),(1,0,1,1)
\end{array}$$

Choose a letter to get a pair of vectors (x, y). $x = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, y = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$. Use $1 + 1 = 0$ in the matrix multiplications to build

$$Mx = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad My = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Step II) Transmission.
Now add an error by switching one entry in each vector:

$$u = Mx + e = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad v = My + f = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Step III) Detect the error e and f. Form

$$Hu = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}, Hv = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} .$$

Now look in which column Hu or Hv is. Assume this column is k. Place 0's everywhere in the vectors e except at the k'th entry where you put 1. For example if Hu is the second column, then put a 1 at the second place. We obtain vectors e and f: $e = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad f = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$.

Step IV) Decode the message.

Use $P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$ to determine $Pe = P \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, Pf = P \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$. In an error-free transmission (Pu, Pv) would give the right result back. Now

$$Pu = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \qquad Pv = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

satisfy $Pu = x + Pe, Pv = y + Pf$. We recover the original message (x, y) and so the letter from

$$x = Pu - Pe = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \qquad y = Pv - Pf = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Assignment: perform the encoding with your initials and check that you got the right letter back. Tell then in one sentence what the essence of this error correction is. In particular, how does the image of the "encoding matrix" M fit with the kernel of the "healing matrix" H?