Lecture 7: Linear transformations

A **transformation** T from a set X to a set Y is a rule, which assigns to every x in X an element $y = T(x)$ in Y. One calls X the **domain** and Y the **codomain**. A transformation is also called a **map** from X to Y. A map T from $\mathbf{R}^m$ to $\mathbf{R}^n$ is called a **linear transformation** if there is a $n \times m$ matrix A such that $T(\vec{x}) = A\vec{x}$.

- To the linear transformation $T(x,y) = (3x+4y, x+5y)$ belongs the matrix $\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$. This transformation maps the two-dimensional plane onto itself.
- $T(x) = -33x$ is a linear transformation from the real line onto itself. The matrix is $A = [-33]$.
- To $T(\vec{x}) = \vec{y} \cdot \vec{x}$ from $\mathbf{R}^3$ to $\mathbf{R}$ belongs the matrix $A = \vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$. This 1×3 matrix is also called a **row vector**. If the codomain is the real axes, one calls the map also a **function**.
- $T(x) = x\vec{y}$ from $\mathbf{R}$ to $\mathbf{R}^3$. $A = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is a 3×1 matrix which is also called a **column vector**. The map defines a line in space.
- $T(x,y,z) = (x,y)$ from $\mathbf{R}^3$ to $\mathbf{R}^2$, A is the 2×3 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The map projects space onto a plane.
- To the map $T(x,y) = (x+y, x-y, 2x-3y)$ belongs the 3×2 matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix}$. The image of the map is a plane in three dimensional space.
- If $T(\vec{x}) = \vec{x}$, then T is called the **identity transformation**.

A transformation T is linear if and only if the following properties are satisfied: $T(\vec{0}) = \vec{0}$ $\quad T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ $\quad T(\lambda\vec{x}) = \lambda T(\vec{x})$

In other words, linear transformations are compatible with addition, scalar multiplication and also respect 0. It does not matter, whether we add two vectors before the transformation or add the transformed vectors.

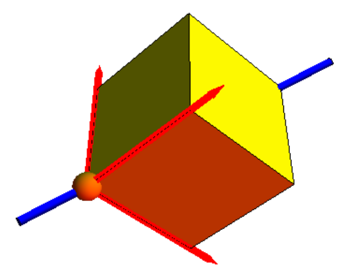
Linear transformations are important in

- geometry (i.e. rotations, dilations, projections or reflections)
- art (i.e. perspective, coordinate transformations),
- computer adided design applications (i.e. projections),
- physics (i.e. Lorentz transformations),
- dynamics (linearizations of general maps are linear maps),
- compression (i.e. using Fourier transform or Wavelet transform),
- error correcting codes (many codes are linear codes),
- probability theory (i.e. Markov processes).
- linear equations (inversion is solving the equation)

A linear transformation $T(x) = Ax$ with $A = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix}$ has the property that the column vector $\vec{v}_1, \vec{v}_i, \vec{v}_n$ are the images of the **standard vectors** $\vec{e}_1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$, $\vec{e}_i = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix}$, and $\vec{e}_n = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

In order to find the matrix of a linear transformation, look at the image of the standard vectors and use those to build the columns of the matrix.

1 Find the matrix belonging to the linear transformation, which rotates a cube around the diagonal $(1,1,1)$ by 120 degrees $(2\pi/3)$.



2 Find the linear transformation, which reflects a vector at the line containing the vector $(1,1,1)$.

If there is a linear transformation S such that $S(T\vec{x}) = \vec{x}$ for every $\vec{x}$, then S is called the **inverse** of T. We will discuss inverse transformations later in more detail.

$A\vec{x} = \vec{b}$ means to invert the linear transformation $\vec{x} \mapsto A\vec{x}$. If the linear system has exactly one solution, then an inverse exists. We will write $\vec{x} = A^{-1}\vec{b}$ and see that the inverse of a linear transformation is again a linear transformation.

3 Otto Bretscher's book contains as a motivation a "code", where the encryption happens with the linear map $T(x, y) = (x + 3y, 2x + 5y)$. It is an variant of a Hill code. The map has the inverse $T^{-1}(x, y) = (-5x + 3y, 2x - y)$.
Assume we know, the other party uses a Bretscher code and can find out that $T(1, 1) = (3, 5)$ and $T(2, 1) = (7, 5)$. Can we reconstruct the code? The problem is to find the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is useful to decode the Hill code in general. If $ax+by = X$ and $cx+dy = Y$, then $x = (dX - bY)/(ad - bc), y = (cX - aY)/(ad - bc)$. This is a linear transformation with matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the corresponding matrix is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$.

"Switch diagonally, negate the wings and scale with a cross".

Homework due February 16, 2011

1

This is Problem 24-40 in Bretscher: Consider the circular face in the accompanying figure. For each of the matrices $A_1, ...A_6$, draw a sketch showing the effect of the linear transformation $T(x) = Ax$ on this face.
$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. $A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $A_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $A_5 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. $A_6 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2 This is problem 50 in Bretscher. A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.
a) King Hiero of Syracuse orders a crown from this goldsmith, with a total mass of 5 kilograms (or 5,000 grams), with the stipulation that the platinum alloy must make up at least 90% of the mass. The goldsmith delivers a beautiful piece, but the king's friend Archimedes has doubts about its purity. While taking a bath, he comes up with a method to check the composition of the crown (famously shouting "Eureka!" in the process, and running to the king's palace naked). Submerging the crown in water, he finds its volume to be 370 cubic centimeters. How much of each alloy went into this piece (by mass)? Is this goldsmith a crook?
b) Find the matrix A that transforms the vector

$$\begin{bmatrix} mass\ of\ platinum\ alloy \\ mass\ of\ silver\ alloy \end{bmatrix}$$

into the vector

$$\begin{bmatrix} totalmass \\ totalvolume \end{bmatrix}$$

for any piece of jewelry this goldsmith makes.
c) Is the matrix A in part (b) invertible? If so, find the its inverse. Use the result to check your answer in part a)

3 In the first week we have seen how to compute the mean and standard deviation of data.
a) Given some data $(x_1, x_2, x_3, ..., x_6)$. Is the transformation from $\mathbf{R}^6 \to \mathbf{R}$ which maps the data to its mean m linear?
b) Is the map which assigns to the data the standard deviation σ a linear map? c) Is the map which assigns to the data the difference $(y_1, y_2, ..., y_6)$ defined by $y_1 = x_1, y_2 = x_2 - x_1, ..., y_6 = x_6 - y_5$ linear? Find its matrix. d) Is the map which assigns to the data the normalized data $(x_1 - m, x_2 - m, ..., x_n - m)$ given by a linear transformation?