Lecture 6: The structure of solutions

Last time we have learned how to row reduce a **matrix**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} .$$

And bring it to so called **row reduced echelon form**. We write rref(A). In this form, we have in each row with nonzero entries a so called **leading one**.

> The number of leading ones in rref(A) is called the **rank** of the matrix.

1 If we have a unique solution to $A\vec{x} = \vec{b}$, then rref(A) is the matrix which has a leading 1 in every column. This matrix is called the **identity matrix**.

> A matrix with one column is also called a **column vector**. The entries of a matrix are denoted by a_{ij}, where i is the row number and j is the column number.

There are two ways how we can look a system of linear equation. It is called the "row picture" or "column picture":
Row picture: each b_i is the dot product of a row vector $\vec{w}_i$ with $\vec{x}$.

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ -\vec{w}_2- \\ \cdots \\ -\vec{w}_n- \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \cdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

Column picture: $\vec{b}$ is a sum of scaled column vectors $\vec{v}_j$.

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b} .$$

2 The system of linear equations

$$\left| \begin{array}{ll} 3x - 4y - 5z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right|$$

is equivalent to $A\vec{x} = \vec{b}$, where A is a **coefficient matrix** and $\vec{x}$ and $\vec{b}$ are **vectors**.

$$A = \begin{bmatrix} 3 & -4 & -5 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} .$$

The **augmented matrix** is

$$B = \left[\begin{array}{ccc|c} 3 & -4 & -5 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right] .$$

In this case, the row vectors of A are
$\vec{w}_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix}$
$\vec{w}_2 = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$
$\vec{w}_3 = \begin{bmatrix} -1 & -1 & 3 \end{bmatrix}$
The column vectors are
$\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ -2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$

The **row picture** tells: $0 = b_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The **column picture** tells:

$$\begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} .$$

A system of linear equations $A\vec{x} = \vec{b}$ with n equations and m unknowns is defined by the $n \times m$ matrix A and the vector $\vec{b}$. The row reduced matrix rref(B) of the augmented matrix $B = [A|b]$ determines the number of solutions of the system $Ax = b$. The **rank** rank(A) of a matrix A is the number of leading ones in rref(A).

> Theorem. For any system of linear equations there are three possibilities:
> • **Consistent with unique solution**: Exactly one solution. There is a leading 1 in each column of A but none in the last column of the augmented matrix B.
> • **Inconsistent with no solutions.** There is a leading 1 in the last column of the augmented matrix B.
> • **Consistent with infinitely many solutions**. There are columns of A without leading 1.

How do we determine in which case we are? It is the rank of A and the rank of the augmented matrix $B = [A|b]$ as well as the number m of columns which determine everything:
If rank(A) = rank(B) = m: there is **exactly 1 solution**.
If rank(A) < rank(B): there are **no solutions**.
If rank(A) = rank(B) < m: there are ∞ **many solutions**.

If we have n equations and n unknowns, it is most likely that we have exactly one solution. But remember Murphy's law "If anything can go wrong, it will go wrong". It happens!

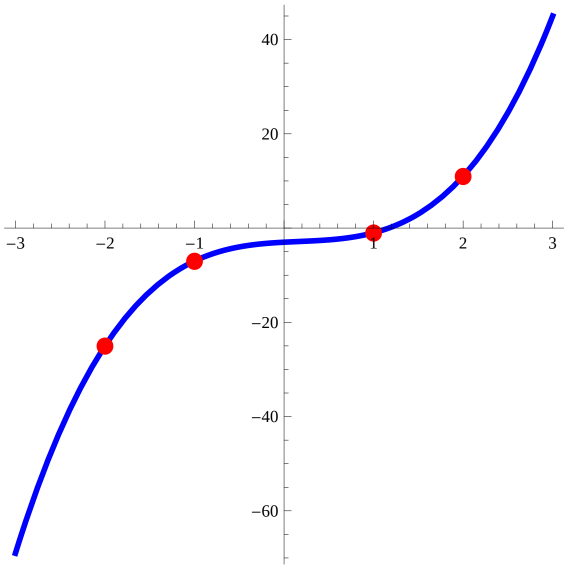
3 What is the probability that we have exactly one solution if we look at all $n \times n$ matrices with entries 1 or 0? You explore this in the homework in the 2×2 case. During the lecture we look at the 3×3 case and higher, using a **Monte Carlo simulation**.

Homework due February 10, 2011

1 We look at the probability space of all 2×2 matrices with matrix entries 0 or 1.
a) What is the probability that the rank of the matrix is 1?
b) What is the probability that the rank of the matrix is 0?
c) What is the probability that the rank of the matrix is 2?

2 Find a cubic equation $f(x) = ax^3 + bx^2 + cx + d = y$ such that the graph of f goes through the 4 points

$$A = (-1, -7), B = (1, -1), C = (2, 11), D = (-2, -25) .$$



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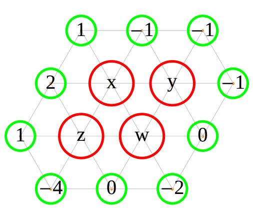

In a Herb garden, the humidity of its soil has the property that at any given point the humidity is the sum of the neighboring humidities. Samples are taken on a hexagonal grid on 14 spots. The humidity at the four locations x, y, z, w is unknown. Solve the equations

$$\left| \begin{array}{rr} x = & y+z+w+2 \\ y = & x+w-3 \\ z = & x+w-1 \\ w = & x+y+z-2 \end{array} \right|$$

using row reduction.