Lecture 5: Gauss-Jordan elimination

We have seen in the last lecture that a system of linear equations like

$$\left| \begin{array}{ll} x+y+z & =3 \\ x-y-z & =5 \\ x+2y-5z & =9 \end{array} \right|$$

can be written in matrix form as $A\vec{x}=\vec{b}$, where A is a **matrix** called **coefficient matrix** and **column vectors** $\vec{x}$ and $\vec{b}$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & -5 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} .$$

The i'th entry $(A\vec{x})_i$ is the dot product of the i'th row of A with $\vec{x}$.

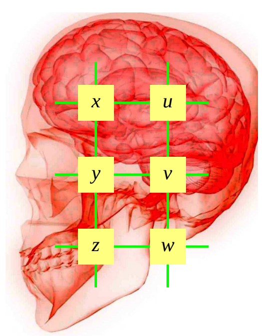
The **augmented matrix** is matrix, where other column has been added. This column contains the vector b. The last column is often separated with horizontal lines for clarity reasons.

$$B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & -1 & 5 \\ 1 & 2 & -5 & 9 \end{array}\right] .$$

We will solve this equation using Gauss-Jordan elimination steps.

1 We aim is to find all the solutions to the system of linear equations

$$\left| \begin{array}{ccccccccccccc} x & & & & & + & u & & & & & = & 3 \\ & & y & & & & & + & v & & & = & 5 \\ & & & & z & & & & & + & w & = & 9 \\ x & + & y & + & z & & & & & & & = & 8 \\ & & & & & & u & + & v & + & w & = & 9 \end{array} \right| .$$



This system appears in **tomography** like magnetic resonance imaging. In this technology, a scanner can measure averages of tissue densities along lines. The task is to compute the actual densities. We first write down the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \\ 1 & 1 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} .$$

Remove the sum of the first three rows from the 4th, then change sign of the 4'th row:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} .$$

Now subtract the 4th row from the last to get a row of zeros, then subtract the 4th row from the first. This is already the row reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -6 \\ 0 & 1 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The first 4 columns have leading 1. The other 2 variables are free variables r, s. We write the row reduced echelon form again as a system and get so the solution:

$$\begin{array}{rcl} x & = & -6+r+s \\ y & = & 5-r \\ z & = & 9-s \\ u & = & 9-r-s \\ v & = & r \\ w & = & s \end{array}$$

There are infinitely many solutions. They are parametrized by 2 free variables.

Gauss-Jordan Elimination is a process, where successive subtraction of multiples of other rows or scaling or swapping operations brings the matrix into **reduced row echelon form**. The elimination process consists of three possible steps. They are called **elementary row operations**:

Swap two rows.
Scale a row.
Subtract a multiple of a row from an other.

The process transfers a given matrix A into a new matrix $\text{rref}(A)$.
The first nonzero element in a row is called a **leading one**. The goal of the Gauss Jordan elimination process is to bring the matrix in a form for which the solution of the equations can be found. Such a matrix is called in **reduced row echelon form**.

1) if a row has nonzero entries, then the first nonzero entry is 1.
2) if a column contains a leading 1, then the other column entries are 0.
3) if a row has a leading 1, then every row above has a leading 1 to the left.

The number of leading 1 in rref(A) is called the rank of A. It is an integer which we will use later.

A remark to the history: The process appeared already in the Chinese manuscript "Jiuzhang Suanshu" the 'Nine Chapters on the Mathematical art'. The manuscript or textbook appeared around 200 BC in the Han dynasty. The German geodesist **Wilhelm Jordan** (1842-1899) applied the Gauss-Jordan method to finding squared errors to work on surveying. (An other "Jordan", the French Mathematician Camille Jordan (1838-1922) worked on linear algebra topics also (Jordan form) and is often mistakenly credited with the Gauss-Jordan process.) **Gauss** developed Gaussian elimination around 1800 and used it to solve least squares problems in celestial mechanics and later in geodesic computations. In 1809, Gauss published the book "Theory of Motion of the Heavenly Bodies" in which he used the method for solving astronomical problems.

2 Find the rank of the following matrix:

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \\ 3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3&3 \\ 4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4 \\ 5&5&5&5&5&5&5&5&5&5&5&5&5&5&5&5&5&5&5&5 \\ 6&6&6&6&6&6&6&6&6&6&6&6&6&6&6&6&6&6&6&6 \\ 7&7&7&7&7&7&7&7&7&7&7&7&7&7&7&7&7&7&7&7 \\ 8&8&8&8&8&8&8&8&8&8&8&8&8&8&8&8&8&8&8&8 \\ 9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9 \end{bmatrix}$$

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3 More challenging is the question: what is the rank of the following matrix?

$$\begin{bmatrix} 1&2&3&4&5&6&7&8 \\ 2&3&4&5&6&7&8&1 \\ 3&4&5&6&7&8&1&2 \\ 4&5&6&7&8&1&2&3 \\ 5&6&7&8&1&2&3&4 \\ 6&7&8&1&2&3&4&5 \\ 7&8&1&2&3&4&5&6 \\ 8&1&2&3&4&5&6&7 \end{bmatrix}.$$

4 What are the possible ranks for a 7×11 matrix?

Homework due February 10, 2011

1 Problem 10 In 1.2 of Bretscher: Find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}$$

2 Problem 20 in 1.2 of Bretscher: We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading l's in the same positions. For example,

$$\begin{bmatrix} 1&2&0 \\ 0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&3&0 \\ 0&0&1 \end{bmatrix}$$

are of the same type. How many types of 2 x 2 matrices in reduced row-echelon form are there?

3 Problem 42 in 1.2 of Bretscher: The accompanying sketch represents a maze of one way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it. What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.