Math 19b: Linear Algebra with Probability **Oliver Knill, Spring 2011**

Lecture 4: Linear equations from Probability

A **linear equation** for finitely many variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \; .$$

Finitely many such equations form a **system of linear equations**. A system of linear equations can be written in matrix form $A\vec{x} = \vec{b}$. Here $\vec{x}$ is the column vector containing the variables, A lists all the coefficients, and $\vec{b}$ is the column vector which lists the numbers to the right.

1 Consider the system

$$\left| \begin{array}{ccccccccccccc} x & + & y & + & z & + & u & + & v & + & w & = 3 \\ & & y & + & z & + & u & + & v & & & = 2 \\ & & & & 2 & + & 2 & & & & & = 4 \end{array} \right| .$$

The are 6 variables and 3 equations. Since we have less equations then unknowns, we expect infinitely many solutions. The system can be written as $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

and $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$.

2 Linear equations appear in probability theory. **Example:** Assume we have a probability space $\Omega = \{x, y, z, w\}$, with four elements. Assume we know the probabilities $P[\{x, y\}] = 4/10$, $P[\{z, w\}] = 6/10$, $P[\{x, z\}] = 7/10$, $P[\{b, d\}] = 3/10$. The question is to find the probabilities $x = P[\{a\}], y = P[\{b\}], z = P[\{c\}], w = P[\{d\}]$.
Answer: This problem leads to a system of equations

$$\begin{array}{cccccccl} x & + & y & + & & & = & 4/10 \\ & & & & z & + & w = & 6/10 \\ x & & & & z & + & = & 7/10 \\ & & y & & & + & w = & 3/10 \end{array}$$

The system can be solved by eliminating the variables. But the system has no unique solution.

3 **Example**. Assume we have two events B_1, B_2 which cover the probability space. We do not know their probabilities. We have two other events A_1, A_2 from which we know $P[A_i]$ and the conditional probabilities $P[A_i|B_j]$. We get the system of equations.

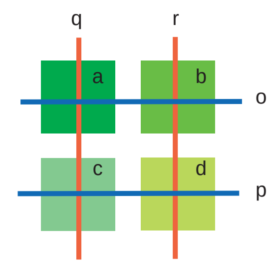
$$\begin{array}{ccccc} P[A_1|B_1]P[B_1] & + & P[A_1|B_2]P[B_2] & = & P[A_1] \\ P[A_2|B_1]P[B_1] & + & P[A_2|B_2]P[B_2] & = & P[A_2] \end{array}$$

Here is a concrete example: Assume the chance that the first kid is a girl is 60% and that the probability to have a boy after a boy is 2/3 and the probability to have a girl after a girl is 2/3 too. What is the probability that the second kid is a girl?
Solution. Let B_1 be the event that the first kid is a boy and let B_2 the event that the first kid is a girl. Assume that for the first kid the probability to have a girl is 60%. But that $P[Firstgirl|Secondgirl] = 2/3$ and $P[Firstboy|Secondboy] = 2/3$. What are the probabilities that the first kid is a boy? This produces a system

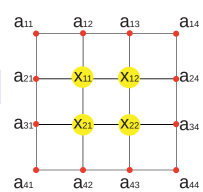
$$\begin{array}{ccccc} 2/3P[B_1] & + & 1/3P[B_2] & = & 6/10 \\ 1/3P[B_1] & + & 2/3P[B_2] & = & 4/10 \end{array}$$

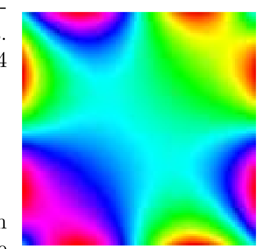
The probabilities are $8/15, 7/15$. There is still a slightly larger probability to have a girl. This example is also at the heart of Markov processes.

4 **Example** Here is a toy example of a problem one has to solve for magnetic resonance imaging (MRI). This technique makes use of the absorb and emission of energy in the radio frequency range of the electromagnetic spectrum.
Assume we have 4 hydrogen atoms, whose nuclei are excited with energy intensity a, b, c, d. We measure the spin echo in 4 different directions. $3 = a+b$, $7 = c+d$, $5 = a+c$ and $5 = b+d$. What is a, b, c, d? Solution: $a = 2, b = 1, c = 3, d = 4$. However, also $a = 0, b = 3, c = 5, d = 2$ solves the problem. This system has not a unique solution even so there are 4 equations and 4 unknowns.



5 We model a drum by a fine net. The heights at each interior node needs the average the heights of the 4 neighboring nodes. The height at the boundary is fixed. With n^2 nodes in the interior, we have to solve a system of n^2 equations. For example, for $n = 2$ (see left), the $n^2 = 4$ equations are
$4x_{11} = x_{21} + x_{12} + x_{21} + x_{12}$,
$4x_{12} = x_{11} + x_{13} + x_{22} + x_{22}$,
$4x_{21} = x_{31} + x_{11} + x_{22} + a_{43}$,
$4x_{22} = x_{12} + x_{21} + a_{43} + a_{34}$.
To the right we see the solution to a problem with $n = 300$, where the computer had to solve a system with $90'000$ variables. This problem is called a Dirichlet problem and has close ties to probability theory too.





6 The last example should show you that linear systems of equations also appear in data fitting even so we do not fit with linear functions. The task is to find a parabola

$$y = ax^2 + bx + c$$

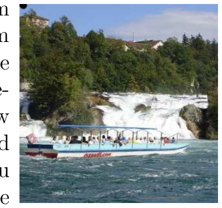
through the points $(1,3)$, $(2,1)$ and $(4,9)$. We have to solve the system

$$\begin{aligned} a+b+c &= 3 \\ 4a+2b+c &= 1 \\ 16a+4b+c &= 9 \end{aligned}$$

The solution is $(2,-8,9)$. The parabola is $y = 2x^2 - 8x + 9$.

Homework due February 10, 2011

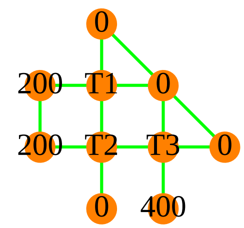
1 Problem 24 in 1.1 of Bretscher): On your next trip to Switzerland, you should take the scenic boat ride from Rheinfall to Rheinau and back. The trip downstream from Rheinfall to Rheinau takes 20 minutes, and the return trip takes 40 minutes; the distance between Rheinfall and Rheinau along the river is 8 kilometers. How fast does the boat travel (relative to the water), and how fast does the river Rhein flow in this area? You may assume both speeds to be constant throughout the journey.



2 (Problem 28 in 1.1 of Bretscher): In a grid of wires, the temperature at exterior mesh points is maintained at constant values as shown in the figure. When the grid is in thermal equilibrium, the temperature at each interior mesh point is the average of the temperatures at the four adjacent points. For example

$$T_2 = (T_3 + T_1 + 200 + 0)/4 \; .$$

Find the temperatures T_1, T_2, T_3 when the grid is in thermal equilibrium.



3 (Problem 46 in 1.1 of Bretscher): A hermit eats only two kinds of food: brown rice and yogurt. The rice contains 3 grams of protein and 30 grams of carbohydrates per serving, while the yogurt contains 12 grams of protein and 20 grams of carbohydrates.
a) If the hermit wants to take in 60 grams of protein and 300 grams of carbohydrates per day, how many servings of each item should he consume?
b) If the hermit wants to take in P grams of protein and C grams of carbohydrates per day, how many servings of each item should he consume?