Lecture 3: Conditional probability

The **conditional probability** of an event A under the condition that the event B takes place is denoted with $\text{P}[A|B]$ and defined to be $\text{P}[A \cap B]/\text{P}[B]$.

1 We throw a coin 3 times. The first 2 times, we have seen head. What is the chance that we get tail the 3th time?
Answer: The probability space Ω consists of all words in the alphabet H, T of length 3. These are $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. The event B is the event that the first 2 cases were head. The event A is the event that the third dice is head.

2 **Problem**: Dave has two kids, one of them is a girl. What is the chance that the other is a girl?
Intuitively one would here give the answer 1/2 because the second event looks independent of the first. However, this initial intuition is misleading and the probability only 1/3.
Solution. We need to introduce the probability space of all possible events

$$\Omega = \{BG, GB, BB, GG\}$$

with $\text{P}[\{BG\}] = \text{P}[\{GB\}] = \text{P}[\{BB\}] = \text{P}[\{GG\}] = 1/4$. Let $B = \{BG, GB, GG\}$ be the event that there is at least one girl and $A = \{GG\}$ the event that both kids are girls. We have

$$\text{P}[A|B] = \frac{\text{P}[A \cap B]}{\text{P}[B]} = \frac{(1/4)}{(3/4)} = \frac{1}{3} .$$

3 You are in the Monty-Hall game show and need to chose from three doors. Behind one door is a car and behind the others are goats. The host knows what is behind the doors. After you open the first door, he opens an other door with a goat. He asks you whether you want to switch. Do you want to?
Answer: Yes, you definitely should switch. You double your chances to win a car:
No switching: The probability space is the set of all possible experiments $\Omega = \{1, 2, 3\}$. You choose a door and win with probability $\boxed{1/3}$. The opening of the host does not affect any more your choice.
Switching: We have the same probability space. If you pick the car, then you lose because the switch will turn this into a goat. If you choose a door with a goat, the host opens the other door with the goat and you win. Since you win in two cases, the probability is $\boxed{2/3}$.

Also here, intuition can lead to **conditional probability traps** and suggest to have a win probability 1/3 in general. Lets use the notion of **conditional probability** to give an other correct argument: the intervention of the host has narrowed the laboratory to $\Omega = \{12, 13, 21, 23, 31, 32\}$ where 21 for example means choosing first door 2 then door 1. Assume the car is behind door 1 (the other cases are similar). The host, who we assume always picks door 2 if you pick 1 with the car (the other case is similar) gives us the **condition** $B = \{13, 21, 31\}$ because the cases 23 and 32 are not possible. The winning event is $A = \{21, 31\}$. The answer to the problem is the conditional probability $\text{P}[A|B] = \text{P}[A \cap B]/\text{P}[B] = \boxed{2/3}$.

If A, B be events in the probability space (Ω, P), then **Bayes rule** holds:

$$\text{P}[A|B] = \frac{\text{P}[B|A] \cdot \text{P}[A]}{\text{P}[B|A] + \text{P}[B|A^c]} .$$

It is a formula for the **conditional probability** $\text{P}[A|B]$ when we know the **unconditional probability** of A and $\text{P}[B|A]$, the **likelihood**. The formula immediately follows from the fact that $\text{P}[B|A] + \text{P}[B|A^c] = \text{P}[B]$.

While the formula followed directly from the definition of conditional probability, it is very useful since it allows us to compute the conditional probability $\text{P}[A|B]$ from the likelihoods $\text{P}[B|A], \text{P}[B|A^c]$. Here is an example:

4 **Problem**. The probability to die in a car accident in a 24 hour period is one in a million. The probability to die in a car accident at night is one in two millions. At night there is 30% traffic. You hear that a relative of yours died in a car accident. What is the chance that the accident took place at night?
Solution. Let B be the event to die in a car accident and A the event to drive at night. We apply the Bayes formula. We know $\text{P}[A \cap B] = \text{P}[B|A] \cdot \text{P}[A] = (1/2000000) \cdot (3/10) = 3/20000000$.

$$\text{P}[A|B] = \frac{\text{P}[A \cap B]}{\text{P}[B]} = (3/20000000)/(1/1000000) = 3/20 .$$

The accident took place at night with a 15 % chance.

A more general version of Bayes rule deals with more than just two possibilities. These possibilities are called $A_1, A_2, ..., A_n$.

Bayes rule: If the disjoint events $A_1, A_2, \dots A_n$ cover the entire laboratory Ω, then

$$\text{P}[A_i|B] = \frac{\text{P}[B|A_i] \cdot \text{P}[A_i]}{\sum_{j=1}^n \text{P}[B|A_j] \cdot \text{P}[A_j]} .$$

Proof: Because the denominator is $\text{P}[B] = \sum_{j=1}^n \text{P}[B|A_j]\text{P}[A_j]$, the Bayes rule just says $\text{P}[A_i|B]\text{P}[B] = \text{P}[B|A_i]\text{P}[A_i]$. But these are by definition both $\text{P}[A_i \cap B]$.

5 **Problem:** A fair dice is rolled first. It gives a random number k from $\{1, 2, 3, 4, 5, 6\}$. Next, a fair coin is tossed k times. Assume, we know that all coins show heads, what is the probability that the score of the dice was equal to 5?
Solution. Let B be the event that all coins are heads and let A_j be the event that the dice showed the number j. The problem is to find $\text{P}[A_5|B]$. We know $\text{P}[B|A_j] = 2^{-j}$. Because the events $A_j, j = 1, \dots, 6$ are disjoint sets in Ω which cover it, we have $\text{P}[B] = \sum_{j=1}^6 \text{P}[B \cap A_j] = \sum_{j=1}^6 \text{P}[B|A_j]\text{P}[A_j] = \sum_{j=1}^6 2^{-j}/6 = (1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64)(1/6) = 21/128$. By Bayes rule,

$$\text{P}[A_5|B] = \frac{\text{P}[B|A_5]\text{P}[A_5]}{(\sum_{j=1}^6 \text{P}[B|A_j]\text{P}[A_j])} = \frac{(1/32)(1/6)}{21/128} = \frac{2}{63} ,$$

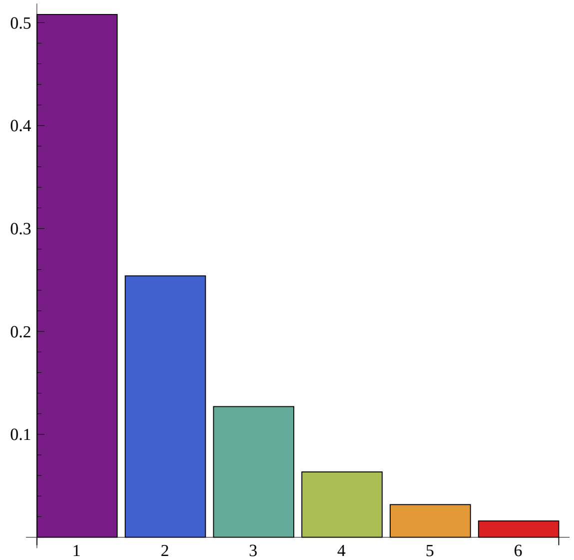

Figure: The conditional probabilities $P[A_j|B]$ in the previous problem. The knowledge that all coins show head makes it more likely to have thrown less coins.

Homework due February 2, 2011

1 Problem 2) in Chapter 3: if the probability that a student is sick at a given day is 1 percent and the probability that a student has an exam at a given day is 5 percent. Suppose that 6 percent of the students with exams go to the infirmary. What is the probability that a student in the infirmary has an exam on a given day?

2 Problem 5) in chapter 3: Suppose that A, B are subsets of a sample space with a probability function P. We know that $\mathrm{P}[A] = 4/5$ and $\mathrm{P}[B] = 3/5$. Explain why $\mathrm{P}[B|A]$ is at least $1/2$.

3 Solve the Monty Hall problem with 4 doors. There are 4 doors with 3 goats and 1 car. What are the winning probabilities in the switching and no-switching cases? You can assume that the host always opens the still closed goat closest to the car.