PROBABILITY THEORY

MATH 154

Midterm Practice

This is a prototype of the format for the take home part. This was essentially the exam from last year. Handwritten. No web-search, no external help, no correspondence, no computer algebra system, no AI. Once you started reading this, no sources, and no notes can be used before submission, except a personally handwritten page (written by yourself before starting the midterm). No discussion about the exam before 3/6/26/2400. Write your name on each page and acknowledge: **"I affirm my awareness of the standards of the Harvard College Honor Code."** Each of the 5 problem has 20 points, each sub-problem is worth 4 points. Your paper has to be submitted in Canvas until Friday 3/6/26/2400 (midnight).

Problem A (20 points): We define here a magic number Ξ ("Xi") that detects all primes! Lets go. Let $\Omega = \mathbb{N} = \{1, 2, 3, \dots\}$ and let $\mathcal{A} = 2^\Omega$ be the set of all subsets and $\mathrm{P}[A] = \sum_{x \in A} 2^{-x}$. For example $P[\{2, 3, 5, 7\}] = 1/2^2 + 1/2^3 + 1/2^5 + 1/2^7$ which in binary expansion is 0.0110101.

a) Verify that the set of sets $\mathcal{I} = \{[a, b) = \{a, a + 1, \dots, b - 1\}\}$ which includes $[a, a) = \emptyset$ is a π-system. A theorem now assures that the measure P defined at first on finite subsets of Ω extends to the σ-algebra $\mathcal{A}$. Who proved this?

b) Check that $\mathrm{P}[\Omega] = 1$. Now look at the set $A = \{2, 4, 6 \dots\}$ of even numbers, which is an event in $\mathcal{A}$. What is its probability $\mathrm{P}[A]$? What is the probability that a natural number is odd in this probability space?

c) Simplify $B = (1 - A) \cdot (1 + A)$ in the Boolean algebra $\mathcal{A}$, where as usual $+$ is the symmetric difference and $\cdot$ is the intersection. How can you characterize the numbers in B if A the set of even numbers?

d) Let P be the set of **prime numbers** and $\Xi = \mathrm{P}[P]$ the probability that a natural number is prime. We measure it to be $\Xi \sim 0.414683....$ Argue that if you know Ξ, you know from any integer, whether it is prime or not!

e) Argue why the irrationality of Ξ implies that there are infinitely many primes. (Ξ is indeed irrational. You do not have to show that as it taps into something we did not cover, but if you see an argument, feel free to include it.)

Problem B (20 points): Theory:
a) What did the **Vitali example** prove? Content and significance.
b) State the **sorority relation** between π and λ systems and σ algebras.
c) State the product formula for the **moment generating functions** for s sum of independent random variables. d) State the product formula for the **characteristic function** of a sum of two independent random variables.
e) Describe the **Lévy inversion formula** in the context of characteristic functions.

Problem C (20 points): Let $([0,1], \mathcal{B}, \mathrm{P})$ be the **Lebesgue probability space** satisfying $\mathrm{P}[[a,b)] = b - a$ on the **π-system** of all half open intervals. Define the events $A = [0,1/2], B = [1/4,1]$ and the random variables $X(x) = x^3, Y(x) = x^5$.
a) Compute $\mathrm{P}[B|A], \mathrm{P}[A|B]$ and verify the **Bayes law** in this case.
b) Compute the **expectations and variance** of both X and Y.
c) What is the **standard deviation** $\sigma[X]$ and $\sigma[Y]$?
d) Find the **covariance** $\mathrm{Cov}[X,Y]$ between X and Y.
e) Compute the **correlation** $\mathrm{Corr}[X,Y] = \mathrm{Cov}[X,Y]/(\sigma[X]\sigma[Y])$.

Problem D (20 points): In this problem, we assume that all random variables are in $\mathcal{L}^2$.
a) Verify that $\mathrm{Var}[X] > 0$ if and only if X is not constant almost everywhere.
b) Give a geometric reason why the identity $\sigma[X - Y] \leq \sigma[X] + \sigma[Y]$ holds.
c) We introduced an angle α between two non-constant random variables by writing $\cos(\alpha) = \mathrm{Corr}[X,Y]$. We were allowed to introduce this angle because of a mathematical inequality $-1 \leq \mathrm{Corr}[X,Y] \leq 1$. What was the name of the inequality which assured this?
d) Prove that for any two random variables $X, Y, Z = X + Y$, the cos-formula holds

$$\mathrm{Var}[X] + \mathrm{Var}[Y] = \mathrm{Var}[Z] - 2\mathrm{Cov}[X,Y] .$$

e) Show that if two random variables are decorrelated then, with $Z = X - Y$ the Pythagorean theorem holds:

$$\mathrm{Var}[X] + \mathrm{Var}[Y] = \mathrm{Var}[Z] .$$

Problem E (20 points): Theorems:

a) Formulate and quickly prove the **Bayes theorem**.

b) State the content of the **Carathéodory theorem**.

c) Give the first part of the **Borel-Cantelli lemma**.

d) Finally write down the second part of the **Borel-Cantelli lemma**. e) What does the **Kolmogorov 0-1 theorem** tell?

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