PROBABILITY THEORY

MATH 154

Take home midterm, Due 3/6/26, midnight

Handwritten. No web-search, no external help, no correspondence, no computer algebra system, no AI. Once you started reading this, no sources, and no notes can be used before submission. You can use a personally handwritten page written by yourself before the midterm. No discussion about the exam before the submission date 3/6/26 midnight. Write your name on each page and acknowledge: **"I affirm my awareness of the standards of the Harvard College Honor Code."**

Problem A (10 points): In this problem, we assume that $(\Omega, \mathcal{A}, \mathrm{P})$ is an arbitrary probability space.
a) Assume that A, B are independent elements in $\mathcal{A}$. Prove that $\mathrm{P}[A+B]$ is determined from $a = \mathrm{P}[A]$ and $b = \mathrm{P}[B]$ by giving a formula of $\mathrm{P}[A+B]$ that only involves a, b.
b) Assume that A, B are independent elements in $\mathcal{A}$. Show that $\mathrm{P}[A \setminus B]$ is determined from $a = \mathrm{P}[A]$ and $b = \mathrm{P}[B]$.
c) Assume A, B are disjoint independent elements in $\mathcal{A}$. Prove that either A or B must be the empty set (up to measure zero).
d) Verify that if $\{A, B, C\}$ are independent sets, then C is independent of $A + B$. Also show that C is independent of $A \cdot B$.
e) Assume that A_n is a sequence of events that satisfy $\sum_{n=1}^{\infty} \mathrm{P}[A_n] = 2026$. What is $\mathrm{P}[\bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n]$?

Problem B (10 points): In this problem a) and b) belong together. Random variables in in c),d),e) are each unrelated to any of the others.
a) Assume the CDF of a random variable X is $F_X(t) = 1 - e^{-5t}$ for $t \geq 0$ and $F_X(t) = 0$ otherwise. What is the PDF $f_X(t)$?
b) What is the expectation and variance of X with the CDF in a)?
c) The **characteristic function** of a random variable X is $\cos(t)$, the characteristic function of Y is $\cos(2t)$. Assuming X, Y are independent. What is the characteristic function of $X + Y$?
d) There is a general formula that determines the CDF $F_X(t)$ of X from $\phi_X(t)$. State it.
e) What is the 4th moment $\mathbb{E}[Z^4]$ of the random variable Z with characteristic function $\phi_Z(t) = \cos(3t)$?

Problem C (10 points): Let $([0,1] \times [0,1], \mathcal{A}, \mathrm{P})$ be the probability space on the unit square Ω. The probability measure satisfies $\mathrm{P}[[a,b) \times [c,d)] = (b-a)(d-c)$ on the π-**system** of all half open squares. Define the events $A = [1/2, 1] \times [0,1], B = [0,1] \times [0,1/3]$ and the random variables $X(x,y) = x^2, Y(x,y) = y$.
a) Compute $\mathrm{P}[B|A], \mathrm{P}[A|B]$ and verify the **Bayes law** in this case.
b) Compute the **expectation**, and **variance** of both X and Y.
c) Find the **covariance** $\mathrm{Cov}[X,Y]$ and **correlation** $\mathrm{Corr}[X,Y]$.
d) Explain why A is in the σ-algebra $\mathcal{A}_X$ generated by X and why B is in the σ algebra $\mathcal{A}_Y$ generated by Y.
e) Explain why X and Y are independent.

Problem D (10 points): In this problem, a, b, λ are positive constants. If X is a Cauchy distributed random variable with probability density $f_X(x) = \frac{1}{\pi(1+\frac{x^2}{a^2})}$, then a computation shows that the characteristic function is $\phi_X(t) = e^{-a|t|}$. The **standard Cauchy distribution** is $a = 1$.
a) Write down the integral which needs to be checked in order to verify that $\phi_X(t)$ is indeed the characteristic function of $f_X(x)$. You do not have to evaluate this integral.
b) Assume Y is a second Cauchy distributed random variable with probability density $f_Y(x) = \frac{1}{\pi(1+\frac{x^2}{b^2})}$. Assuming that X, Y are independent, argue carefully why $X + Y$ is a Cauchy distributed random variable.
c) Prove that if X is Cauchy distributed, then $Y = \lambda X$ is Cauchy distributed if $\lambda > 0$.
d) How do the characteristic functions of X and λX relate?
e) Verify that if X, Y are standard Cauchy distributed random variables that are independent, then $(X+Y)/2$ is a standard Cauchy distributed random variable.

Problem E (10 points): We look at some Paradoxa and Theorems:
a) What did the **Bertrand paradox** show us? State content and significance.
b) State the content of the **Carathéodory theorem**.
c) What structure is the smallest π-system that contains a λ-system $\mathcal{A}$.
d) Explain the Simpsons paradox in words.
e) We verify a more sophisticated **Bayes formula**. Recall that a partition of Ω is a disjoint union of events, whose union is Ω. Prove the following: "Given a finite partition $\{A_1, \dots, A_n\}$ in $\mathcal{A}$ and $B \in \mathcal{A}$ with $\mathrm{P}[B] > 0$, then for every $1 \leq k \leq n$, we have

$$\mathrm{P}[A_k|B] = \frac{\mathrm{P}[B|A_k]\mathrm{P}[A_k]}{\sum_{j=1}^n \mathrm{P}[B|A_j]\mathrm{P}[A_j]} .$$