PROBABILITY THEORY

MATH 154

Unit 20: De Moivre-Laplace and Poisson

20.1. The central limit theorem is applied whenever we have to analyze data. If X_i are IID random variables in $\mathcal{L}^2$ with mean m and standard deviation σ, we can ask for the probability that S_n/n is within $\epsilon/\sqrt{n}$ to the mean m. The central limit helps us validate data by observing averages of experiments. Here is the application version:

Theorem 1. *The probability that S_n/n deviates more than $t\sigma/\sqrt{n}$ from the mean $m = \mathrm{E}[X]$ can for large n be estimated by*

$$\frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} \, dx \; .$$

Proof. Let $m = \mathrm{E}[X]$ denote the mean of X_k and σ the standard deviation. Denote by X a random variable which has the standard normal distribution $N(0,1)$. Write $X_n \sim Y_n$ if $X_n \to^d Y_n$ in distribution. By the central limit theorem

$$\frac{S_n - nm}{\sqrt{n}\sigma} \sim X \; .$$

Dividing both nominator and denominator by n gives $\frac{\sqrt{n}}{\sigma}(\frac{S_n}{n} - m) \sim X$ so that as distributions

$$\frac{S_n}{n} - m \sim X \frac{\sigma}{\sqrt{n}} \; .$$

But this means that we can estimate the deviation as $1 - F_{N(0,1)}(t)$, which is the expression we see in the theorem. $\square$

20.2. The term $\sigma/\sqrt{n}$ is called the **standard error**. The value t is the **z-score** and $1 - F_{N(0,1)}(t)$ the p-value. We will discuss a concrete case in class.

20.3. Let us look at the case of coin tossing. These are independent $\{0,1\}$-valued random variables with win probability $p \in (0,1)$ was historically the starting point for the central limit theorem. The sum S_n has a **Binomial distribution** $B(n,p)$ of mean np and variance $np(1-p)$. As we have just seen, the fact that

$$\lim_{n\to\infty} \mathrm{P}[\frac{(S_n - np)}{\sqrt{np(1-p)}} \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} \, dy$$

is a consequence of the central limit theorem. It has already been proven by de Moivre in 1730 in the case $p = 1/2$ and for general $p \in (0,1)$ by Laplace in 1812.

Theorem 2 (DeMoivre-Laplace limit theorem)**.** *If S_n have the Binomial distribution $B(n,p)$, then S_n^* converges in distribution to $N(0,1)$.*

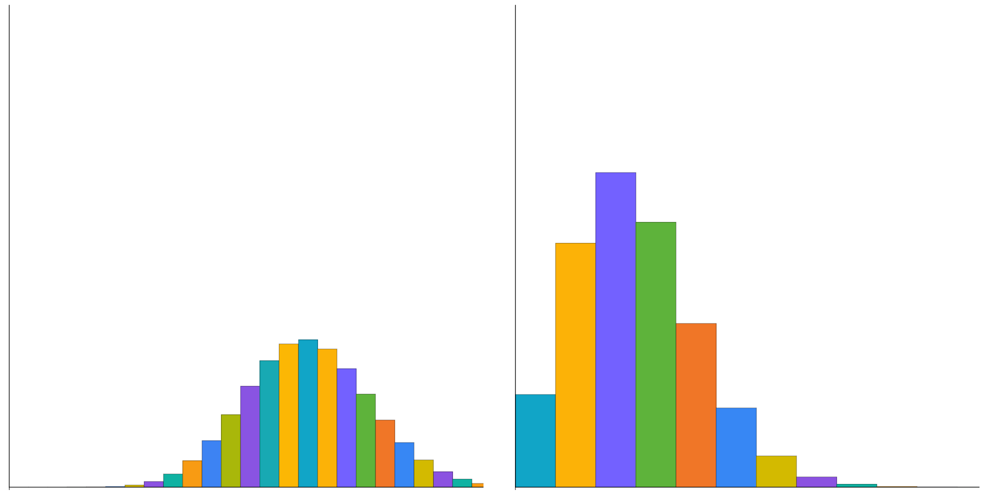

Figure 1. Binomial distribution $B(n, 0.3)$ to the left for $n = 50$. The Binomial distribution $B(n, 1/n)$ to the right for $n = 50$. The left will for $n \to \infty$ when rescaled will converge to the normal distribution. The right will converge to the Poisson distribution.

20.4. The **Poisson distribution** P_λ supported on $\mathbb{N} = \{0, 1, 2, , 3 \dots\}$ is defined as

$$\mathrm{P}[X = k] = e^{-\lambda} \frac{\lambda^k}{k!} .$$

Random variables with such distribution describe waiting times. Its moment generating function is

$$M_X(t) = \sum_{k=0}^{\infty} \mathrm{P}[X = k] e^{tk} = e^{\lambda(1-e^t)} .$$

That the Poisson distribution is natural follows from:

Theorem 3 (Poisson limit theorem)**.** *Let X_n be a $B(n, p_n)$-distributed and suppose $np_n \to \lambda$. Then X_n converges in distribution to a random variable X with Poisson distribution with parameter λ.*

20.5. For the proof we need the last time used **compound interest statement** telling that if $a_n \to 0, b_n a_n \to c$ implies $(1 + a_n)^{b_n} \to e^c$. It is related to $(1 + c/n)^{1/n} \to e^c$.

Proof. We have to show that $\mathrm{P}[X_n = k] \to \mathrm{P}[X = k]$ for each fixed $k \in \mathbb{N}$.

$$\begin{aligned} \mathrm{P}[X_n = k] &= \binom{n}{k} p_n^k (1 - p_n)^{n-k} \\ &= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} p_n^k (1 - p_n)^{n-k} \\ &\sim \frac{1}{k!} (np_n)^k (1 - \frac{np_n}{n})^{n-k} \to \frac{\lambda^k}{k!} e^{-\lambda} . \end{aligned}$$

$\square$

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