

PROBABILITY THEORY

MATH 154

Unit 17: Recurrence

17.1. A map $T : \Omega \rightarrow \Omega$ on a probability space (Ω, \mathcal{A}, P) is **measurable** if $T^{-1}(A) \in \mathcal{A}$ for every $A \in \mathcal{A}$. It is **measure preserving** if $P[T^{-1}(A)] = P[A]$, for all $A \in \mathcal{A}$. If T is invertible and T^{-1} is also measure preserving, then T is called an **automorphism**. Automorphisms form a group $\text{Aut}(\Omega)$. On a finite probability space with $P[A] = |A|/|\Omega|$, automorphisms are permutations. Ergodic means cyclic.

17.2. Probability spaces with measure preserving maps as **morphisms** form a **category**. As before we use the notation $T^n(\omega) = T(T(\dots T(\omega)))$ for the n 'th iterate. Poincaré proved in 1890:

Theorem 1 (Poincaré recurrence). *Given an automorphism T and $A \in \mathcal{A}$ with $P[A] > 0$, then $P[T^n(A) \cap A] > 0$ for some $n \leq 1/P[A]$.*

Proof. Otherwise, the events $\{A_k = T^k(A)\}$ are distinct and $1 = P[\Omega] \geq P[\bigcup_{k=1}^n A_k] = \sum_{k=1}^n P[A_k] = nP[A] > 1$. Contradiction. \square

17.3. Recall that T is called **ergodic** if every fixed point $T^{-1}(A) = A$ has probability 0 or 1. In other words, $P[A + T^{-1}(A)] = 0 \Rightarrow P[A]^2 = P[A]$. A single random variable X defines a stochastic process $X_n(\omega) = X(T^n(\omega))$.

17.4. We have seen last time that ergodicity is the same than having only constant random variables invariant under T . If T is ergodic and $X = 1_A$ then the Birkhoff ergodic theorem assures that the number S_n of hits of $T^n(\omega)$ in A satisfies $S_n/n \rightarrow E[X] = P[A]$ for almost all ω . ”**space average agrees with time average**”.

17.5. Write the circle $\mathbb{R}/(2\pi\mathbb{Z})$ using complex numbers as $\Omega = \{|z| = 1\} \subset \mathbb{C}$. It is equipped with the probability measure $P[\text{Arg}(z) \in [a, b]] = (b - a)/(2\pi)$ for $0 < a < b < 2\pi$ and the Borel σ -algebra \mathcal{A} . If $w = e^{2\pi i\alpha}$ is a complex number of length 1, then the rotation $T(z) = wz$ defines a measure preserving transformation on (Ω, \mathcal{A}, P) . It is invertible with inverse $T^{-1}(z) = z/w$. This system is called the **Kronecker system**. We can also write it as $\theta \rightarrow \theta + \alpha \pmod{2\pi}$.

Theorem 2. *For irrational α , the Kronecker system is ergodic.*

Proof. Write a random variable $X \in \mathcal{L}^2(\mathbb{T})$ on Ω as a Fourier series. ¹ $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ with $a_n = E[z^n f]$. Decompose $f = f_0 + f_+ + f_-$ into $f_+ = \sum_{n=1}^{\infty} a_n z^n$ analytic in $|z| < 1$, and $f_- = \sum_{n=1}^{\infty} a_n z^{-n}$ analytic in $|z| > 1$ and constant f_0 . The same decomposition for $f(T(z)) = \sum_{n=-\infty}^{\infty} a_n w^n z^n$ shows that $f_+ = \sum_{n=1}^{\infty} a_n z^n =$

¹We switch notation and write $f = X$ to be closer to calculus.

$\sum_{n=1}^{\infty} a_n w^n z^n$. But these are the Taylor expansions of $f_+ = f_+(T)$ and so $a_n = a_n w^n$. Because $w^n \neq 1$ for irrational α , we deduce that $a_n = 0$ for $n \geq 1$. Similarly, one derives $a_n = 0$ for $n \leq -1$. Therefore $f(z) = a_0$ is constant. \square

17.6. It follows that for every number $x \in [0, 1]$ and every irrational α and every $\epsilon > 0$, there exists n such that $|n\alpha - x| < \epsilon$.

17.7. The transformation $T(z) = z^2$ on the same probability space as in the previous example is also measure preserving. Note that $P[T(A)] = 2P[A]$ but $P[T^{-1}(A)] = P[A]$ for all $A \in \mathcal{A}$. The map is measure preserving, but it is **not invertible**.

Theorem 3. *The squaring transformation $T(z) = z^2$ on the circle is ergodic.*

Proof. A Fourier argument shows it again. Again, T honors the decomposition of f into three analytic functions $f = f_- + f_0 + f_+$ so that $f(T(z)) = \sum_{n=-\infty}^{\infty} a_n z^{2n} = \sum_{n=-\infty}^{\infty} a_n z^n$ implies $\sum_{n=1}^{\infty} a_n z^{2n} = \sum_{n=1}^{\infty} a_n z^n$. Comparing Taylor coefficients of this identity for analytic functions shows $a_n = 0$ for odd n because the left hand side has zero Taylor coefficients for odd powers of z . But because for even $n = 2^l k$ with odd k , we have $a_n = a_{2^l k} = a_{2^{l-1} k} = \dots = a_k = 0$, all coefficients $a_k = 0$ for $k \geq 1$. Similarly, one sees $a_k = 0$ for $k \leq -1$. \square

17.8. The single angle random variable $X(x) = \arg(x)$ on Ω , produces a sequence of random variables $X_n(x) = X(T^n(x))$. The squaring system is conjugated to the shift $S(x)_n = x_{n+1}$ on the product probability space $(\{0, 1\}^{\mathbb{N}}, \mathcal{B}, P)$. The conjugating map is $\phi(x) = e^{2\pi x_i / 2^i} \in \Omega$. We have $\phi(S(x)) = T(\phi(x))$.

17.9. If A is an event with $P[A] > 0$ and T is a measure preserving automorphism, define a new transformation $T_A(x) = T^{n_A(x)}(x)$, where $n_A(x)$ is the **return time**, the smallest $n > 0$ with $T^n(x) \in A$. By Poincaré recurrence, the random variable $n_A(x)$ is finite for almost all $x \in A$. We can look at the **conditional probability space** $(A, \mathcal{A} \cap A, P/P[A])$ and the **induced dynamical system** T_A .

Theorem 4. *T_A is an automorphism of $(A, \mathcal{A} \cap A, P/P[A])$. It is ergodic if T is ergodic.*

Proof. (i) T_A is measure preserving. Decompose $A = \bigcup_{k=1}^{\infty} A_k$ with $A_k = \{n_A = k\}$. Now $T_A(x) = T^k(x)$ for $x \in A_k$. Given $B \in \mathcal{A} \cap A$ define $B_k = B \cap A_k$ so that $B = \bigcup_k B_k$. $P[T_A^{-1}(B)] = P[T_A^{-1}(\bigcup_k B_k)] = P[\bigcup_k T_A^{-1}(B_k)] = P[\bigcup_k T^{-k} B_k] = \sum_k P[T^{-k}(B_k)] = \sum_k P[B_k] = P[\bigcup_k B_k] = P[B]$. (ii) If T is ergodic then T_A is ergodic. Proof: use contradiction. If $T_A(B) = B$ has $P[B] < P[A]$ then $C = \bigcup_k T^{-k}(B)$ is T invariant with $P[C] < 1$ and T is not ergodic. \square

17.10. Note that it is possible that T_A is ergodic but T is not ergodic. Kakutani noticed that if $\bigcup_k T^k A = \Omega$, then T_A is ergodic if and only if T is ergodic.

17.11. If T is a measure preserving transformation on a probability space, we can look at the longer incidences $A \cap T^{-n}(A) \cap \dots \cap T^{-kn}(A)$. Fürstenberg showed in 1977:

Theorem 5 (Multiple recurrence theorem). *For any $A \in \mathcal{A}$ with $P[A] > 0$, there exists n such that $P[A \cap T^{-n}(A) \cap \dots \cap T^{-kn}(A) \cap A] > 0$.*

It implies the **van der Waerden theorem** telling that any r -coloring of the integers \mathbb{N} contains a color which contains arbitrary large arithmetic progressions.