Unit 9: Tail algebras

9.1. Given a family $\{\mathcal{A}_i\}_{i\in I}$ of σ-subalgebras of $\mathcal{A}$. For any nonempty set $J \subset I$, let $\mathcal{A}_J := \bigvee_{j\in J} \mathcal{A}_j$ be the σ-algebra generated by $\bigcup_{j\in J} \mathcal{A}_j$. Define also $\mathcal{A}_\emptyset = \{\emptyset, \Omega\}$. The **tail σ-algebra** $\mathcal{T}$ of $\{\mathcal{A}\}_{i\in I}$ is defined as $\mathcal{T} = \bigcap_{J\subset I, J \text{ finite}} \mathcal{A}_{J^c}$, where $\mathcal{J}^c = \mathcal{I} \setminus \mathcal{J}$. An algebra $\mathcal{A}$ is called **trivial** if $\mathrm{P}[A] = 0$ or $\mathrm{P}[A] = 1$ for all $A \in \mathcal{A}$. There are σ algebras with infinitely many elements that are trivial: the σ algebra of countable or cocountable sets in the Lebesgue space $([0,1], \mathcal{B}, dx)$ is an example because every countable set has probability 0 and every cocountable set has probability 1.

Theorem 1 (Kolmogorov's $0-1$ law). *If $\{\mathcal{A}_i\}_{i\in I}$ are independent σ-algebras, then the tail σ-algebra $\mathcal{T}$ is trivial.*

Proof. (i) Assume $F, G \subset I$ are disjoint sets. Then $\mathcal{A}_F$ and $\mathcal{A}_G$ are independent.
Proof. Define for $H \subset I$ the π-system $\mathcal{I}_H = \{A \in \mathcal{A} \mid A = \bigcap_{i\in K} A_i, K \subset_f H, A_i \in \mathcal{A}_i\}$.
The π-systems $\mathcal{I}_F$ and $\mathcal{I}_G$ are independent and generate the σ-algebras $\mathcal{A}_F$ and $\mathcal{A}_G$.
(ii) Especially: $\mathcal{A}_J$ is independent of $\mathcal{A}_{J^c}$ for every $J \subset I$.
(iii) $\mathcal{T}$ is independent of $\mathcal{A}_I$.
Proof. $\mathcal{T} = \bigcap_{J\subset_f I} \mathcal{A}_{J^c}$ is independent of any $\mathcal{A}_K$ for $K \subset_f I$. It is therefore independent to the π-system $\mathcal{I}_I$ which generates $\mathcal{A}_I$.
(iv) $\mathcal{T}$ is a sub-σ-algebra of $\mathcal{A}_I$. Therefore $\mathcal{T}$ is independent of itself which implies that it is P-trivial. □

9.2. Example: Let X_n be a sequence of independent random variables and let $A = \{\omega \in \Omega \mid \sum_{n=1}^\infty X_n \text{ converges }\}$. Then $\mathrm{P}[A] = 0$ or $\mathrm{P}[A] = 1$. Proof. Because $\sum_{n=1}^\infty X_n$ converges if and only if $Y_n = \sum_{k=n}^\infty X_k$ converges, we have $A \in \sigma(A_n, A_{n+1} \dots)$. Therefore, A is in $\mathcal{T}$, the tail σ- algebra defined by the independent σ-algebras $\mathcal{A}_n = \sigma(X_n)$. If for example, if X_n takes values $\pm 1/n$, each with probability $1/2$, then $\mathrm{P}[A] = 0$. If X_n takes values $\pm 1/n^2$ each with probability $1/2$, then $\mathrm{P}[A] = 1$. The decision whether $\mathrm{P}[A] = 0$ or $\mathrm{P}[A] = 1$ is related to the convergence or divergence of a series. This will be discussed later again in the context of limit theorems.

9.3. Example: Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of Ω. The set $A_\infty := \limsup_{n\to\infty} A_n = \bigcap_{m=1}^\infty \bigcup_{n\geq m} A_n$ consists of the set $\{\omega \in \Omega\}$ such that $\omega \in A_n$ for infinitely many $n \in \mathbb{N}$. The set A_∞ is contained in the tail σ-algebra of $\mathcal{A}_n = \{\emptyset, A_n, A_n^c, \Omega\}$. It follows from Kolmogorov's $0-1$ law that $\mathrm{P}[A_\infty] \in \{0,1\}$ if $A_n \in \mathcal{A}$ and $\{A_n\}$ are P-independent.

Theorem 2 (Borel-Cantelli Lemma). *Take any sequence $A_n \in \mathcal{A}$.*
a) $\sum_{n\in\mathbb{N}} \mathrm{P}[A_n] < \infty \Rightarrow \mathrm{P}[A_\infty] = 0$ *always holds.*
b) $\sum_{n\in\mathbb{N}} \mathrm{P}[A_n] = \infty \Rightarrow \mathrm{P}[A_\infty] = 1$, *if A_n are independent.*

Proof. a) $\mathrm{P}[A_\infty] = \lim_{n\to\infty} \mathrm{P}[\bigcup_{k\geq n} A_k] \leq \lim_{n\to\infty} \sum_{k\geq n} \mathrm{P}[A_k] = 0$.
b) For every integer $n \in \mathbb{N}$,

$$\mathrm{P}[\bigcap_{k\geq n} A_k^c] = \prod_{k\geq n} \mathrm{P}[A_k^c] = \prod_{k\geq n} (1-\mathrm{P}[A_k]) \leq \prod_{k\geq n} e^{-\mathrm{P}[A_k]} = e^{-\sum_{k\geq n} \mathrm{P}[A_k]} .$$

From

$$\mathrm{P}[A_\infty^c] = \mathrm{P}[\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k^c] \leq \sum_{n\in\mathbb{N}} \mathrm{P}[\bigcap_{k\geq n} A_k^c] = 0$$

follows $\mathrm{P}[A_\infty^c] = 0$. □

9.4. The following example illustrates that independence is necessary in the part b) of the Borel-Cantelli lemma: take the probability space $([0,1], \mathcal{B}, \mathrm{P})$, where $\mathrm{P} = \lambda$ is the Lebesgue measure on the Borel σ-algebra $\mathcal{B}$ of $[0,1]$. For $A_n = [0, 1/n]$ we get $A_\infty = \emptyset$ and so $\mathrm{P}[A_\infty] = 0$. But because $\mathrm{P}[A_n] = 1/n$ we have $\sum_{n=1}^\infty \mathrm{P}[A_n] = \sum_{n=1}^\infty \frac{1}{n} = \infty$ because the **harmonic series** $\sum_{n=1}^\infty 1/n$ diverges: $\sum_{n=1}^R \frac{1}{n} \geq \int_1^R \frac{1}{x}\, dx = \log(R)$.

9.5. Writing a novel amounts to enter a sequence of N symbols into a computer. For "Hamlet", Shakespeare had to enter $N = 180'000$ characters. Pop-culture [1] imagines a monkey typing randomly for an indefinite time, producing a random text. Call A_n the event that Monkey types Hamlet on the interval $[nN, , ...nN+N]$. These sets A_n are all independent and have probability 26^{-N}. Since $\sum_n A_n = \infty$ Borel-Cantelli assures that the even appears infinitely often. Reality produces constraints like that monkeys like humans live less than $4 * 10^9$ seconds but mathematicians do not care about such things. Their ideas live for ever!

9.6. A nice application of Borel-Cantelli are **percolation problems**. If we take an infinite connected network = graph and delete each bond=edge randomly with probability p, then there will be a threshold p_c such that for $p > p_c$ the network has no infinite cluster and for $p < p_c$ there is an infinite cluster. The event "there is an infinite cluster" is in the tail σ algebra of a set of σ algebras $\mathcal{A}_e = \{0, 1, N_e, N_e^c\}$, where N_e are the set of networks for which bond e is active and N_e^c the set of networks for which edge e is broken. The index set $J = E$ enumerates all the edges of the network and the tail σ-algebra $\mathcal{T}$ **consists of all events that do not change if a finite part of the network is altered.**

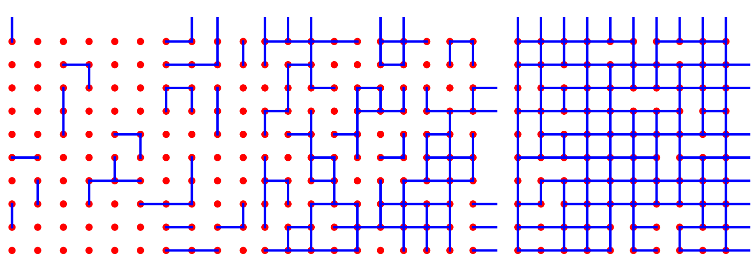

FIGURE 1. The bond percolation threshold in dimension 2 is known to be $p = 1/2$. We see random lattice networks with $p = 0.2, p = 0.5, p = 0.8$.

[1] First appearing in Feller's book from 1950