PROBABILITY THEORY

MATH 154

Unit 8: Characteristic functions

8.1. Given $X \in \mathcal{L}$, its **characteristic function** is a complex-valued function on $\mathbb{R}$ defined as $\phi_X(t) = \mathrm{E}[e^{itX}]$. Compare this with the **moment generating function** $M_X(t) = \mathrm{E}[e^{tX}]$. It is important to note that the characteristic function is better behaved because it always exists. The moment generating function needed boundedness. For a Cauchy distributed random variable for example, the moment generating function does not exist as not even $\mathrm{E}[X^2]$ exists. The characteristic function however does exist as $e^{itX} = \cos(tX) + i\sin(tX)$ is a bounded complex-valued random variable.

8.2. If $F = F_X$ is the distribution function of X and $\mu = \mu_X$ is its law, the characteristic function of X is also known as the Fourier-Stieltjes transform because $\phi_X(t) = \int_{\mathbb{R}} e^{itx}\, dF(x) = \int_{\mathbb{R}} e^{itx}\, d\mu(x)$. If F has a derivative f, the PDF, then ϕ_X is called the **Fourier transform** of the density function f_X: $\phi_X(t) = \int_{\mathbb{R}} e^{itx}\, f_X(x)\, dx$.

8.3. Example: For a random variable with density $f_X(x) = x^m/(m+1)$ on $\Omega = [0,1]$ the characteristic function is

$$\phi_X(t) = \int_0^1 e^{itx} x^m\, dx/(m+1) = \frac{m!(1 - e^{it} e_m(-it))}{(-it)^{1+m}(m+1)} ,$$

where $e_n(x) = \sum_{k=0}^n x^k/(k!)$ is the n'th **partial exponential function**.

8.4.

Theorem 1 (Lévy)**.** *The characteristic function ϕ_X determines the distribution of X.*

There are explicit formulas. If a, b are points of continuity of F, then

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\, dt .$$

If one or both of the end points have mass, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\, dt = \mu[(a,b)] + \frac{1}{2}\mu[\{a\}] + \frac{1}{2}\mu[\{b\}] .$$

Proof. Because a distribution function F has only countably many points of discontinuities, it is enough to determine $F(b) - F(a)$ in terms of ϕ if a and b are continuity points of F. The verification of the **Lévy formula** is then a computation. For continuous distributions with density $F_X' = f_X$ is the inverse formula for the Fourier transform: $f_X(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ita}\phi_X(t)\, dt$ so that $F_X(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ita}}{-it}\phi_X(t)\, dt$. This proves the inversion formula if a and b are points of continuity.

The general formula needs only to be verified when μ is a point measure at the boundary of the interval. By linearity, one can assume μ is located on a single point b with $p = \mathrm{P}[X = b] > 0$. The Fourier transform of the Dirac measure $p\delta_b$ is $\phi_X(t) = pe^{itb}$. The claim reduces to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it}\, pe^{itb}\, dt = \frac{p}{2}$$

which is equivalent to the claim $\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{itc}-1}{it}\, dt = \pi$ for $c > 0$. Because the imaginary part is zero for every R by symmetry, only

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(tc)}{t}\, dt = \pi$$

remains. The verification of this integral is a prototype computation in residue calculus.

□

We say that a sequence X_n of random variables converges weakly to X if and only if its characteristic functions converge point wise: $\phi_{X_n}(x) \to \phi_X$. Here is a table of characteristic functions (CF) $\phi_X(t) = \mathrm{E}[e^{itX}]$ and moment generating functions (MGF) $M_X(t) = \mathrm{E}[e^{tX}]$ for some familiar random variables:

Distribution	Parameter	CF	MGF
Normal	$m \in \mathbb{R}$, $\sigma^2 > 0$	$e^{mit-\sigma^2 t^2/2}$	$e^{mt+\sigma^2 t^2/2}$
$N(0,1)$		$e^{-t^2/2}$	$e^{t^2/2}$
Uniform	$[-a, a]$	$\sin(at)/(at)$	$\sinh(at)/(at)$
Exponential	$\lambda > 0$	$\lambda/(\lambda - it)$	$\lambda/(\lambda - t)$
Binomial	$n \geq 1$, $p \in [0,1]$	$(1 - p + pe^{it})^n$	$(1 - p + pe^{t})^n$
Poisson	$\lambda > 0$, λ	$e^{\lambda(e^{it}-1)}$	$e^{\lambda(e^{t}-1)}$
Geometric	$p \in (0,1)$	$\frac{p}{(1-(1-p)e^{it}}$	$\frac{p}{(1-(1-p)e^{t}}$
Cauchy	$m \in \mathbb{R}$, $b > 0$	$e^{imt-\|t\|}$	$e^{mt-\|t\|}$

Characteristic functions become especially useful, if one deals with independent random variables. Their characteristic functions multiply:

Theorem 2. *Given independent random variables X, Y, then $\phi_X(t)\phi_Y(t) = \phi_{X+Y}(t)$.*

Proof. Since X_j are independent, we get for any set of complex valued continuous functions g_j, for which $\mathrm{E}[g_j(X_j)]$ exists:

$$\mathrm{E}[\prod_{j=1}^{n} g_j(X_j)] = \prod_{j=1}^{n} \mathrm{E}[g_j(X_j)]\ .$$

Proof: This follows almost immediately from the definition of independence since one can check it first for functions $g_j = 1_{A_j}$, where A_j are $\sigma(X_j)$ measurable functions for which $g_j(X_j)g_k(X_k) = 1_{A_j \cap A_k}$ and

$$\mathrm{E}[g_j(X_j)g_k(X_k)] = \mathrm{P}[A_j]\mathrm{P}[A_k] = \mathrm{E}[g_j(X_j)]\mathrm{E}[g_k(X_k)]\ ,$$

then for step functions by linearity and then by approximation for arbitrary continuous functions.
If we put $g_j(x) = \exp(ix)$, the proposition is proved. □