Unit 3: Algebras

3.1. If Ω is a set, a set $\mathcal{A}$ of subsets of Ω is an **algebra** if it is closed under **intersection** $\cap$ and **symmetric difference** Δ and if $\Omega \in \mathcal{A}$. The algebra of sets behaves like the algebra of integers. We just have to think about $\Delta = +$ as addition and $\cap = \cdot$ as multiplication and $\emptyset$ as 0 and Ω as 1. **Commutativity**, **associativity** and **distributivity** can be seen as logical identities. We can also visualize them as **Venn diagrams**. Figure 2) shows associativity $(A \cdot B) \cdot C = A \cdot (B \cdot C)$. It just encodes the set of elements in Ω which are in all of the sets. The algebra encodes logical thinking rules that usually are taken for granted. Boolean algebra includes also Boolean logic like the "tertium non datur" $A + A^c = \Omega$.

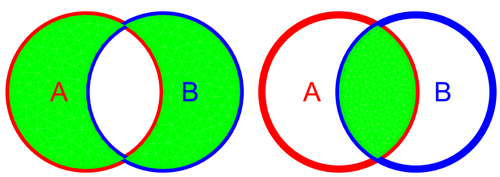

FIGURE 1. Venn diagram of the addition and multiplication in the algebra.

3.2. Any algebra of sets always contains the **empty set** $0 = \emptyset$, which is part of the algebra because it is $\Omega + \Omega$. The empty set is also called **zero** because $0 + A = A$ for every A. The set Ω plays the role of the 1 because $1 \cdot A = A$. The algebra is also called a **Boolean algebra** because $A + A = 0$ and so $-A = A$. We can form other set operations like the union $A \cup B = AB + A + B = 1 + (1 + A)(1 + B)$ and the set difference $A \setminus B = B + AB$ and the **complement** $A^c = A + 1$. A Boolean algebra is a **commutative ring** with 1. Besides the laws $A + A = 0$ and $A^2 = A$, we have in particular $1 + 1 = 0$.

3.3. A set I is called **countable** if there is a bijection from I to the counting numbers $\mathbb{N} = \{1, 2, 3 \dots\}$. Every countable set of sets can therefore be written as a sequence of sets $\{A_1, A_2, \dots\}$. The **Hilbert hotel** pop-culture picture is the result that $\mathbb{N}$ and $2\mathbb{N} = \{2, 4, 6, 8, 10, \dots\}$ have the same cardinality. We can also count the rationals $\mathbb{Q}$, as seen in class. We can not count the numbers in the interval $[0, 1]$ as Cantor showed in his famous diagonal argument: just assume to have an enumeration and construct from this a new number that is different from each of the numbers.

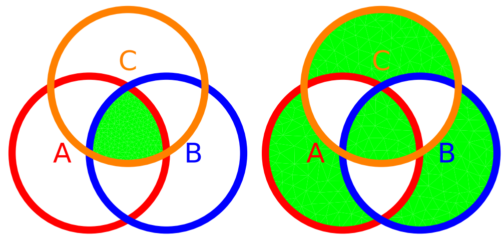

FIGURE 2. The Venn diagrams of multiplicative associativity $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ and additive associativity $A + (B + C) = (A + B) + C$.

3.4. An algebra is called **σ-algebra** if it is closed under the formation of countable unions. A pair $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is a σ-algebra on Ω is also called a **measurable space**. Formally, $A_n \in \mathcal{A} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$. This implies that countable intersections are in $\mathcal{A}$: $\bigcap_n A_n = 1 - (\bigcup_n (1 - A_n))$.

3.5. Examples: 1) Given Ω, $\mathcal{A} = \{\emptyset, \Omega\}$ is a σ-algebra: the **trivial** σ-algebra.
2) GIven Ω, then $\mathcal{A} = 2^\Omega = \{A \subset \Omega\}$ is the largest σ-algebra one can define on Ω.
3) A finite set of pairwise disjoint sets $A_1, A_2, \dots, A_n$ of Ω satisfying $\bigcup_j A_j = \Omega$ is a **finite partition** of Ω. It generates the finite σ-algebra $\mathcal{B} = \{A = \bigcup_{j \in J} A_j\ \}$, where J runs over all subsets of $\{1, \dots, n\}$. The elements $\{A_1, \dots, A_n\}$ are the **atoms** of $\mathcal{B}$.
4) For any $\mathcal{C} \subset 2^\Omega$, the intersection of all σ-algebras containing $\mathcal{C}$ is the σ-algebra **generated** by $\mathcal{C}$.
5) The intersection of arbitrary many σ algebras is a σ algebra.
6) A **topology** $\mathcal{O}$ on Ω, is a set of sets that contains $\emptyset, \Omega$ and is closed under finite intersections and arbitrary unions. The σ-algebra generated by $\mathcal{O}$ is the **Borel σ algebra** of $\mathcal{O}$

3.6. Write $A_n \nearrow A$ if $A_n \subset A_{n+1}$ and $\bigcup_n A_n = A$. We say A is a **limit**. $\mathcal{A}$ is called a **π-system**, if $\mathcal{A}$ is closed under intersections. $\mathcal{A}$ is called a **λ-system** or Dynkin system if $\mathcal{A}$ contains Ω, is is closed under complements and closed under limits. Neither π systems nor λ systems need to be algebras. The following is called **π-λ theorem**:

Theorem 1. *$\mathcal{A}$) is a σ-algebra $\Leftrightarrow$ $\mathcal{A}$ is a π-system and a λ-system.*

Proof. " $\Rightarrow$ ": Just check that $A \setminus B = A \cup B + B = A + AB$.
" $\Leftarrow$ ": if $\mathcal{A}$ is both a π-system and a λ system. Given $A, B \in \mathcal{A}$. By definition we know that $A^c = \Omega \setminus A, B^c = \Omega \setminus B$ is in $\mathcal{A}$. The π-system property implies that $A \cup B = \Omega \setminus (A^c \cap B^c) \in \mathcal{A}$. We also have $A + B = A \cup B \setminus A \cap B$ in $\mathcal{A}$. Given a sequence $A_n \in \mathcal{A}$. Define $B_n = \bigcup_{k=1}^n A_k \in \mathcal{A}$ and $A = \bigcup_n A_n$. Because $B_n \nearrow A$ we know that A is a limit and that $A \in \mathcal{A}$. This finishes the proof that $\mathcal{A}$ is a σ-algebra. $\square$

3.7. Examples: 1) A topology is a π-system but not a λ-system.
2) A filter is a π system by definition but not a λ-system.
3) A finite abstract simplicial complex is not π system in general.
4) The set of all half open intervals $[a, b)$ with $a < b$ is a π system.
5) The set $\mathcal{A} = \{\emptyset, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{1,2,3,4\} = \Omega\}$ is a λ system that is not a π-system.