PROBABILITY THEORY

MATH 154

Unit 2: Classical challenges

The Simpson paradox

2.1. The **Simpson paradox** was popularized 30 years ago with the following example: there are two type of kidney stone surgeries: open surgery is (A) or minimal invasion surgery (B). Overall, B has 83 percent success rate, while A has 78 percent success rate. $\boxed{\text{B is better}}$ Now split the samples into two groups depending on whether the patient had "small stones" or "big stones". For small stones, the success rate of A is 93 percent, while the success rate for B is 87 percent. For large stones, the success rate for A is 73 percent, while the success rate for B is 69 percent. If we split up the sample, $\boxed{\text{A is better}}$. The example illustrate how easy it is to manipulate data without really cheating as both conclusions are correct. You want to know this trick if you aim to go into politics, journalism or marketing ...

The Bertrand paradox

2.2. What is the probability of the event A that a random line on the unit disc intersects it with a length $\geq \sqrt{3}$, the length of the inscribed equilateral triangle? Looking at all lines through a boundary points can be given by an angle in $[0, \pi]$. In the interval $(\pi/3, 2\pi/3$ the line segment is longer, we see that $\boxed{\mathrm{P}[A] = 1/3}$. By looking at all points perpendicular to a fixed diameter we conclude $\boxed{\mathrm{P}[A] = 1/2}$. By comparing at the area of the region that the center of the chord hits gives probability is $\boxed{1/4}$. Which of the arguments is correct?

The Monty-Hall paradox

2.3. You're on a game show and you are given a choice of three doors. Behind one door is a car and behind the others are goats. You pick a door-say No. 1 - and the host, who knows what's behind the doors, opens another door-say, No. 3-which has a goat. (In all games, he opens a door to reveal a goat). The host tells: "Do you want to pick door No. 2?" (In all games he always offers an option to switch). Is it to your advantage to switch your choice?

The Banach Tarski paradox

2.4. It is possible to write the ball $X = \{x^2 + y^2 + z^2 \leq 1/9\}$ as a disjoint union of 5 sets $X = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ and rotate and translate the sets in Ω to sets B_1, B_2, B_3, B_4, B_5 such that $B_1 \cup B_2 \cup B_3 = \{(x-1/2)^2 + y^2 + z^2 \leq 1/9\} = X - (1/2, 0, 0)$ and $B_4 \cup B_5 = \{(x+1/2)^2 + y^2 + z^2 \leq 1/9\} = X + (1/2, 0, 0)$. By cutting up a ball, translating and rotating the pieces, the ball has doubled in size.

The Petersburg paradox

2.5. Assume you pay an entrance fee c for a game and that you win 2^T, where T is the number of times, the casino flips a coin until "head" appears. For example, if the sequence of coin experiments would give "tail, tail, tail, head", you would win $2^3 - c = 8 - c$, the win minus the entrance fee. For which c is the game fair? We can compute the expectation as $\sum_{k=1}^{\infty} 2^k \mathrm{P}[T = k] = \sum_{k=1}^{\infty} 1 = \infty$. But nobody would agree to pay even an entrance fee $c = 20$. The event $T = 20$ is so improbable that it never occurs in the life-time of a person.

The Martingale paradox

2.6. Here is a bullet proof **martingale strategy** in roulette: bet c dollars on red. If you win, stop, if you lose, bet $2c$ dollars on red. If you win, stop. If you lose, bet $4c$ dollars on red. Keep doubling the bet. Eventually after n steps, red will occur and you will win $2^n c - (c + 2c + \cdots + 2^{n-1} c) = c$ dollars. This example motivates the concept of martingales. Why can this foolproof strategy not used?

The Birthday paradox

2.7. Intuition can be misleading. We often see coincidences and label them "impossible" while it is just combinatorics that explain them. This is the most famous example: there are 365 days in a year, so that that it appears that we appear to need a larger group of people to expect a Birthday collision. It turns out that already for a group with 23 people, the probability that two have the same birthday is larger than 1/2. Coincidences happen more frequently than expected.

The entropy paradox

2.8. Basic fundamental laws of physics are reversible, whether it is quantum mechanics, classical mechanics, general relativity, the standard model for elementary particles. A container $\Omega = A \cup B$ has two separate rooms and filled with gas so that the probability to be in A is $p = 1/3$ and to be in B is $q = 2/3$. The pressure in chamber B is twice as big as in part A. The entropy is defined as $S = -p \log(p) - q \log(q) = 0.636....$ Now open the door. The gas will go to an equilibrium with $p = 1/2, q = 1/2$ and entropy $= 0.6931...$ The entropy has increased. The "second law of thermodynamics" claims that entropy can only increase and in general increases. If we look at the dynamics of the n particles in the room, they move deterministically. The dynamics is complicated but the **recurrence theorem of Poincare** tells that after some time, we will again have the situation where the pressure in chamber A gives $p = 1/3$ and in chamber B is $q = 2/3$. Entropy obviously has decreased. We have violated the "second law".