PROBABILITY THEORY

MATH 154

Homework 7

Strong Law and Birkhoff

Problem 7.1: We construct an example of a stochastic process that satisfies the weak law but not the strong law. Let $(\Omega = [0,1]^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}}, \mathrm{P}^{\mathbb{N}})$ denote the standard product Lebesgue probability space. Consider for $n \geq 1$ the sequence $a_n = 1/(n\log(n))$. Define $X_n(x) = n1_{[0,a_n/2](x_n)} - n1_{[1-a_n/2,1](x_n)}$, a sequence of random variables taking only values in $\{n, -n, 0\}$.
a) Check that X_n is a sequence of independent random variables of zero mean and variance $n/(\log(n))$.
b) Check that the proof of the weak law of large numbers still works so that $\mathrm{P}[S_n/n \geq \epsilon] \to 0$.
c) Verify that $\sum_n \mathrm{P}[\{X_n = n\}]$ diverges and conclude that with probability 1, we have $|S_n/n| \geq 1/2$ infinitely many often.
d) Conclude that X_n does not satisfy the strong law of large numbers.

Problem 7.2: A transformation $T : \Omega \to \Omega$ on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ is called **measure preserving** if for $A \in \mathcal{A}$, the set $T^{-1}(A)$ is in $\mathcal{A}$ and $\mathrm{P}[T^{-1}(A)] = \mathrm{P}[A]$. Given a random variable X, we have a sequence of random variables $X_n(\omega) = X(T^n x)$.
a) Assume $(\mathbb{T}^2 = \mathbb{R}/\mathbb{Z}, \mathcal{A}, dx)$ is the 1-torus. Verify that $T(x) = 2x \bmod 1$ is measure preserving. For example $T(0.7) = 0.4$. (We will see later that X_n are independent random variables.)
b) Define the transformation $T(x,y) = (2x - y + \sin(2\pi x), x)$ on the two dimensional torus $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$, equipped with the product probability space $dxdy$ (area). Verify that T is measure preserving, and that T is invertible.
c) The von Neumann-Kakutani system is also called the adding machine or unit translation on the dyadic group of integers. It is a measure preserving map on the interval $[0,1]$ which is on $[0,1/2]$ defined as $f(x) = x + 1/2$ on $[1/2, 1/2+1/4]$ defined as $f(x) = x + 1/4$ and on $[1/2+1/4+1/8]$ is defined as $f(x) = x + 1/8$ etc. Verify that it is measure preserving.

Problem 7.3: Given a real number α, define the transformation T on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{T}$ as $T(x) = x + \alpha \bmod 1$. A continuous function $f : \mathbb{T} \to \mathbb{R}$ defines so a sequence of random variables $X_n(x) = f(T^n(x)) = f(x + n\alpha)$.
a) If there exists a continuous function g such that $f(x) = g(x+\alpha) - g(x)$, we call f a coboundary). What can you say about the growth rate of S_n, if f is a coboundary?
b) The sum S_n is also known as a Weyl sum. Assume f is continuous with $\int_0^1 f(x)\, dx = 0$ and that α is irrational. What does the Birkhoff ergodic theorem say about S_n/n?
c) Assume α is irrational. Are the random variables X_n independent? Are the random variables decorrelated? Can you use the strong law of large numbers to estimate S_n? Can you use the weak law of large numbers to estimate S_n?
d) Look up what happens if α has the Diophantine property $|\alpha - p/q| \geq 1/q^2$ for all rational numbers p/q. (An example is if α is the golden mean.) There is a result that assures that S_n stays bounded in this Diophantine case if f is continuous. Find that result and state it.

Problem 7.4: Use the notes to write down the proof of the maximal ergodic theorem of Hopf. Make sure you understand every step.

Problem 7.5: Use the notes to write down the proof of the Birkhoff ergodic theorem. Make sure you understand every step.

Oliver Knill, knill@math.harvard.edu, Math 154, Spring, 2026