PROBABILITY THEORY

MATH 154

Homework 6

Stochastic Convergence

Problem 6.1: a) Is there for $1 \leq p < \infty$ a relation between L^p convergence and convergence almost everywhere?
b) Is there a relation between L^∞ convergence and convergence almost everywhere?
c) Is there a relation between complete convergence and L^p convergence for $p < \infty$?
d) Is there a relation between complete convergence and L^∞ convergence?

Problem 6.2: Consider the random variables $X_n(x) = \cos(nx)$ on $[-\pi, \pi], \mathcal{B}, dx/(2\pi)$.
a) By writing $\cos(nx) = \mathrm{Re}(e^{inx})$ and using a geometric series, verify that

$$S_n(x) = \frac{2\sin((n+1/2)x)}{\sin(x/2)} - 1 \ .$$

This is $D_n(x) - 1$, where $D_n(x)$ is called the Dirichlet kernel.
b) Verify that $||S_n(x)||_1 + 1 \geq \frac{2}{\pi}\log(2n+1)$.
A proof can be found on the Wikipedia page on the Dirichlet kernel. Write it down in your words.
c) First recollect from class why the assumptions of the weak law are satisfied and restate the conclusion of that theorem about S_n/n. This should verify that $S_n/n \to 0$ in $\mathcal{L}^1$ and so in probability.
d) Given a continuous even function f with $E[f] = 0$, the expectation $a_n = E[fX_n]$ is called the n'th Fourier coefficient and $g(x) = \sum_n a_n X_n(x)$ is the cos-Fourier series of n. The formula $\sum_{n=1}^\infty a_n^2 = ||f||_2$ is called Parceval's identity. What geometric condition does assure it and what famous geometric theorem does it generalize?

Problem 6.3: a) Give an example of a sequence of random variables $X_n \to X$ for which we have convergence in probability but not complete convergence.
b) Give an example of a sequence of random variables X_n, where $X_n \to X$ in probability but where $X_n \to X$ in $\mathcal{L}^1$ does not happen.
c) Assume $X_n \to X$ in $\mathcal{L}^3$? For which p can one say $X_n \to X$ in $\mathcal{L}^p$?
d) Give an example of a sequence of random variables where $X_n \to X$ in $\mathcal{L}^1$ but where the convergence is not in $\mathcal{L}^2$.

Law of Large numbers

Problem 6.4: The n'th Chebyshev polynomial is defined as $X_n = T_n(x) = \cos(n \arccos(x))$. We have $T_n(\cos(t) = \cos(nt)$.
a) Verify that $T_n(x)$ is a polynomial of degree n and write down $T_n(x)$ for $n = 1, 2, 3, 4$.
b) We look at $T_n(x)$ as a random variable on the probability space $(\Omega = [-1, 1], \mathcal{B}, \mathrm{P} = \frac{1}{\pi\sqrt{1-x^2}})$. Check that the later indeed is a probability space.
c) Demonstrate (by showing all conditions) that you can use the weak law of large numbers to establish that $(1/n)S_n$ converges in probability to 0.

Problem 6.5: Let $X_n(\omega)$ be the n'th binary digit of $\omega \in [0, 1]$.
a) Investigate the convergence $\frac{S_n}{n} \to m$ in probability.
b) Verify that S_n has the Binomial distribution $p_k = \begin{pmatrix} n \\ k \end{pmatrix} 1/2^n$.
c) Show directly and then use the weak law to see that $S_n/n \to 0$ in probability.
d) Verify that $S_n/\sqrt{n}$ does not go to zero in L^2.
e) Verify also that $S_n/\sqrt{n}$ does not converge to 0 in distribution.