PROBABILITY THEORY

MATH 154

Homework 3

Random Variables

Problem 3.1: The circle $x^2 + y^2 = 1$ can be described using angles $\Omega = [0, 2\pi)$ and equipped with the Lebesgue σ algebra $\mathcal{A}$ and the measure $P[[a,b)] = (b-a)/(2\pi)$. We have a nice probability space $(\Omega, \mathcal{A}, \mathrm{P})$. Consider the random variable $X(\omega) = y = \sin(\omega)$.
a) Compute the mean m variance and standard deviation of X directly like $m = \mathrm{E}[X]$ and $\mathrm{Var}[X] = \mathrm{E}[X^2] - E[X]^2$.
b) Compute the skewness $E[(X-m)^3]/\sigma^3$ and kurtosis $\mathrm{E}[(X-m)^4]/\sigma^4$.
c) Find the cumulative distribution function CDF $F_X(s) = \mathrm{P}[X(\omega) \leq s]$.
d) The probability density function (PDF) f_X of X is supported on $[-1, 1]$. Find it.
e) Compute $m = \mathrm{E}[X]$,$\mathrm{Var}[X]$, and σ from the PDF.

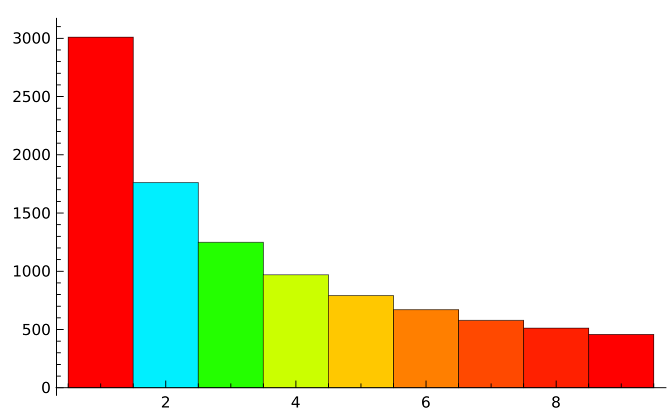

Figure 1. The Benford distribution for the first significant digit. It is computed with Histogram[Table[First[IntegerDigits[2^n]], {n, 1, 10000}], 10]

Problem 3.2: **Benford's law** deals with the statistics of the first significant digit in data. Simon Newcomb found the law in 1881 and Frank Benford made significant progress to understand it in 1938. The distribution appears also in naturally occurring sequences. For example, if you look at the first digit of the sequence 2^n then the first significant digit k appears with probability $p_k = \log_{10}(1 + 1/k)$. The digit 1 for example occurs with about $\log_{10}(2) = 0.30$ which is 30 percent.
a) What is its expectation and variance of the distribution?
b) Verify that the sequence 2^n produces this distribution.

Problem 3.3: The **Gamma distribution** with shape $\alpha > 0$ and rate $\lambda > 0$ has support on $[0, \infty)$. It is used in econometrics. The probability density function is

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha} .$$

a) What distribution do we get in the case $\alpha = 1$?
b) Verify that f satisfies the properties of a PDF.
c) Compute the expectation $\mathrm{E}[X]$ and variance $\mathrm{Var}[X]$.
d) Compute the moment generating function $M_X(t)$.
e) Why is a Gamma distributed random variable in $\mathcal{L}^p$ for all p?

Problem 3.4: Verify that for $\theta > 0$ the **Maxwell distribution**

$$f(x) = \frac{4}{\sqrt{\pi}} \theta^{3/2} x^2 e^{-\theta x^2}$$

is a PDF of a probability distribution on $\mathbb{R}^+ = [0, \infty)$. This distribution can model the speed distribution of molecules in thermal equilibrium. Now compute its expectation $\mathrm{E}[X] = \int_0^\infty x f(x)\, dx$.

Problem 3.5: For a **centered Cauchy distributed** random variable, the probability density is $(\frac{1}{\pi}/(1 + x^2)$. As seen in class you can generate random variables with this distribution. Define $X(x) = x$ on $(\Omega = \mathbb{R}, \mathcal{B}, \mathrm{P} = f(x)dx)$.
a) Check that the random variable X is not in $\mathcal{L}^1$.
b) Look up the definition of convergence "in the sense of Cauchy" and verify that the expectation of the distribution in this generalized sense.
c) What can you say about the variance and higher moments or moment generating function of a Cauchy distributed random variable?
d) Why again does the Mathematica command Cot(Pi $*$ Random[]) generate Cauchy distributed random variables?