

# PROBABILITY THEORY

MATH 154

## Homework 2

### PROBABILITY SPACES

The axioms of a probability space  $(\Omega, \mathcal{A}, P)$  are:

1)  $\mathcal{A}$  is a  $\sigma$ -**algebra**: it is a Boolean ring on which any set operation can be done finitely or countably many times:

- (i)  $(\mathcal{A}, +, \cdot, 0 = \emptyset, 1 = \Omega)$  is a Boolean ring.
- (ii)  $A_n \in \mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$ .

As we have seen in class, you can assume that all other set operations are allowed, like  $A^c = \Omega \setminus A = 1 + A$ ,  $A \cup B = (A^c \cap B^c)^c$  or  $A \cup B = A + B + AB$ .

2)  $P$  is a **probability measure**: it is non-negative, normalized and  $\sigma$ -additive.

- (i)  $P[A] \geq 0$
- (ii)  $P[\Omega] = 1$
- (iii)  $A_n \in \mathcal{A}$  disjoint  $\Rightarrow P[\bigcup_n A_n] = \sum_n P[A_n]$

**Problem 2.1:** Verify the following properties from the axioms:

- a)  $P[\emptyset] = 0$ .
- b)  $A \subset B \Rightarrow P[A] \leq P[B]$ .
- c)  $P[\bigcup_n A_n] \leq \sum_n P[A_n]$ .
- d)  $P[A^c] = 1 - P[A]$ .
- e)  $0 \leq P[A] \leq 1$ .
- f)  $A_1 \subset A_2 \subset \dots$  with  $A_n \in \mathcal{A}$  then  $P[\bigcup_{n=1}^{\infty} A_n] = \lim_{n \rightarrow \infty} P[A_n]$ .

**Problem 2.2:** Let  $\Omega$  be a set. Let  $\mathcal{A}$  be the set of countable or co-countable subsets of  $\Omega$ .

- a) Verify that  $\mathcal{A}$  satisfies all the ring axioms of a Boolean algebra.
- b) Verify that  $\mathcal{A}$  is a  $\pi$ -system.
- c) Verify that  $\mathcal{A}$  is a  $\lambda$ -system.
- d) Verify that  $\mathcal{A}$  is a  $\sigma$  algebra without using the theorem of Lecture 3.
- e) Verify that  $\mathcal{A}$  is the smallest  $\sigma$ -algebra containing the cofinite topology.

**Problem 2.3:** Let  $\Omega = [0, 1]^2$ . Let  $\mathcal{I} = \{[a, b) \times [c, d)\}$  denote the set of all left-bottom closed right-top open rectangles.

- a) Verify that this is a  $\pi$ -system.
- b) Verify that  $P[a, b) \times [c, d) = (d - c)(b - a)$  is a probability measure on this  $\pi$  system.
- c) Why can the measure  $P$  be extended to the smallest  $\sigma$ -algebra containing  $\mathcal{I}$ ?
- d) Under which conditions are two elements in  $\mathcal{I}$  independent?

**Problem 2.4:** Verify the following properties. The first four are known as **Keynes postulates**, the fifth is called **Bayes Theorem**.

- 1)  $P[A|B] \geq 0$ .
- 2)  $P[A|A] = 1$ .
- 3)  $P[A|B] + P[A^c|B] = 1$ .
- 4)  $P[A \cap B|C] = P[A|C] \cdot P[B|A \cap C]$ .
- 5)  $P[A|B] = P[B|A]P[A]/P[B]$ .

**Problem 2.5:** Prove the  $\Pi\Sigma\Lambda$  **sorority theorem** in the text. It states "The smallest  $\lambda$ -system  $\mathcal{A}$  containing a  $\pi$ -system  $\mathcal{I}$  is the smallest  $\sigma$  algebra containing  $\mathcal{I}$ ."



FIGURE 1. To the left an example of a  $\Pi\Sigma\Lambda$  chapter (in this case Oxford MS). To the right, a brooch from BU in the shape of a Marguerite daisy (or  $A \cap B$  when intersecting two sets in a Venn Diagram) also in the order of the mathematical order  $\Pi\Lambda\Sigma$ : to check that we have a  $\sigma$ -algebra, we have to check it is a  $\pi$ -system and a  $\lambda$ -system.