PROBABILITY THEORY

MATH 154

Homework 1

Probability

Problem 1.1: a) What is the probability that a random point (x, y, z) in the unit cube $[-1, 1]^3$ is in the ball $x^2 + y^2 + z^2 \leq 1$?
b) What is the probability that a point in the ball is in the octahedron $|x| + |y| + |z| \leq 1$?
c) What is the probability that $x_1^2 + x_2^2 + \cdots + x_{26}^2 \leq 1$ if the point $x = (x_1, \ldots, x_{26})$ is a random point in the cube $[-1, 1]^{26}$?
d) In which dimension n is the volume of the unit sphere maximal?

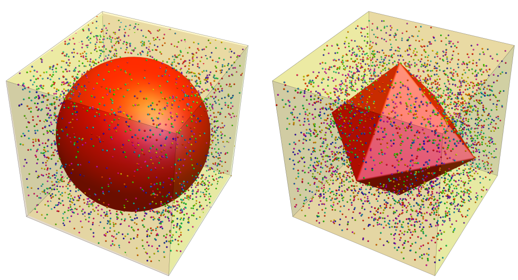

FIGURE 1. What is the probability to hit the ball? What is the probability to hit the octahedron?

Problem 1.2: The card game "set" contains $81 = 3^4$ cards. Each card has one of 3 colors, one of 3 numbers, one of 3 shapes and one of 3 shades. It so models so the 4-dimensional vector space $\mathbb{Z}_3^4$ which is also called the field $GF(81)$. A collection of three cards is called a "set", if in each of the 4 categories, all three properties either agree or are all different. You randomly pick 3 cards from the 81. What is the probability to draw a set?

Problem 1.3: The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ has the reciprocal $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)/n^s$ where $\mu(n)$ is the **Moebius function** (which is 0 if n has a square factor and $(-1)^k$ if n has k different prime factors). Assume you are told that $\mu(n)$ is random enough so that the law of iterated log holds. Prove the **Riemann hypothesis**!

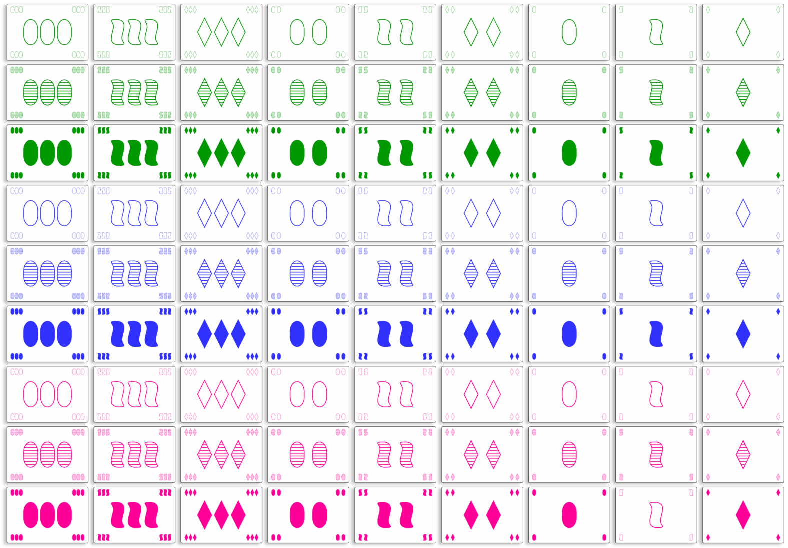

FIGURE 2. The game of set visualizes a 4 dimensional vector space

Problem 1.4: a) Alex has three kids, and one of them is a girl. What is the probability that Alex has three girls?
b) Alex has three kids of different age and the oldest is a girl. What is the probability that Alex has three girls?
c) There are three boxes: a box containing two gold coins, a box containing two silver coins, and a box containing one gold coin and one silver coin. The three boxes are shuffled. You pick one box and pick a random coin from it. You notice it to be gold. What is the probability that the other coin from the same box is gold?

Problem 1.5: Box A is open and contains thousand dollars. Box B is closed and contains either nothing or 1 Million dollars, depending on what the majority of intelligence on earth would pick with the knowledge given here. If the majority chooses [A and B], then B contains nothing. If the majority should chose [B], then B contains a million dollars. You know this and decide whether to take both boxes (and so at least 1000 dollars) or or only box B (with the possibility to leave empty handed or become rich). No other information is available to you. You can discuss with others but your decision should be your own and not be based on collusion! (We want to keep your decision to be independent from decisions of others but allow theoretical arguments to be discussed). Report your choice: [A and B] or [B] and give a reason for your choice, submit it in your HW and also send it to Oliver (*knill@math.harvard.edu*) who will take majority vote to decide whether you got 10^3 or 10^6 dollars - or nothing!